**Math 302**
**Homework 9**

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**Problem 1:** Recall de Morgan Theorem: $(A \cup B)^c = A^c \cap B^c$ for any pair of sets $A$ and $B$.

(a) Use the principle of mathematical induction to show that

$$(A_1 \cup A_2 \cup \ldots . \cup A_n)^c = A_1^c \cap A_2^c \cap \ldots \cap A_n^c$$

for any finite $n \in \mathbf{N}$.

(b) Explain **why** the induction cannot be used to conclude

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

(c) Is the statement in (b) true? If so, write the appropriate proof without using induction.

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**Problem 2:** Prove that any natural power of the golden number $\varphi$ can be expressed via Fibonacci numbers as follows:

$$\varphi^n = \varphi F_n + F_{n-1}$$

Recall: $\varphi = \dfrac{\sqrt{5}+1}{2}$. Also $F_1=1$, $F_2=1$, $F_3=2$ and $F_{n+2} = F_{n+1} + F_n$.

**Problem 3:** Show this curious property of Fibonacci numbers:

$$F_{n-1} \cdot F_{n+1} = (F_n)^2 + (-1)^n$$

**Problem 4:** Prove these properties of Fibonacci numbers:

(i) $F_{3n}$ is even for any natural $n$.
(ii) $F_{4n}$ is divisible by 3.

Guess a general property. Can you prove it?

**Problem 5.** Show that for any natural $n$ the following formula holds:

$$(F_1)^2 + (F_2)^2 + (F_3)^2 + \ldots + (F_n)^2 = F_n F_{n+1}$$

(Discovered by Lukas, 1897). Clearly, first check for a few initial cases.

**Problem 6 (for fun).** (a) A generalized Fibonacci sequence is defined by the usual recurrence, $G_{n+2} = G_{n+1}+G_n$, but starting with two arbitrary natural numbers $G_1=a$, $G_2=b$. For instance choosing a=2 and b=5 we get:

**2**, **5**, 7, 12, 19, 31, 50, ...

Show that the ratio of two consecutive numbers of this sequence $G_{n+1}/G_n$ approaches the golden mean $\varphi = (1+\sqrt{5})/2$.

(b) Calculate the first few entries of $G_i$ for general a and b, and then look...

**Problem 7 (for fun).** In a certain right triangle the sides form a geometric sequence. Find the sides of this triangle. [Hint: you can make the shortest side 1. Why?]. This triangle has been actually built in enormous size – where?