Localization

Definition 1: Let Γ be a multiplicative commutative group. We shall say that an **ordering** is defined in Γ if we are given a subset S of Γ closed under multiplication such that Γ is the disjoint union of S, the unit element 1 and the set S^{-1} consisting of all inverses of elements of S.

Definition 2: Let P be a prime ideal in any ring R and let $D = R - P$. By definition of a primr ideal D is multiplicatively closed. It is easy to see that the set $D^{-1}R$ is a ring (Dummit and Foote, page 262). The ring $D^{-1}R$ in this case is called the **ring of fractions** of R with respect to P or the **localization** of R at P. It is denoted by R_P. Every element in R but not in P is a unit in R_P.

Example: Let $R = \boldsymbol{Z}$ and $P = (p)$ is a prime ideal. Then

$$\boldsymbol{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbf{Q} : p \not| b \right\} \subseteq \mathbf{Q}$$

and every integer b not divisible by p is a unit.

Definition 3: A ring R is called a **local ring** if it is commutative and has a unique maximal ideal. If R is a local ring and $\boldsymbol{m}$ is it's maximal ideal and $x \in R$ but $x \notin \boldsymbol{m}$ then x is a unit (otherwise x generates a proper ideal not contained in $\boldsymbol{m}$, which is impossible).

Example: Recall the field $\mathbf{Q}_p$ of p-adic numbers. The ring of integers of $\mathbf{Q}_p$ denoted by $\mathbf{Z}_p$ was defined as

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \leq 1\}$$

$\mathbf{Z}_p$ is a local ring with the maximal ideal

$$\mathbf{m} = \{x \in \mathbf{Q}_p : |x|_p < 1\}.$$

Definition 4:(**Valuation and Valuation Ring**) Let K be a field. An absolute value on K is a real valued function $x \mapsto |x|$ on K satisfying the following three properties:

(1) $|x| \geq 0$. $|x| = 0$ iff $x = 0$.

(2) $\forall x, y \in K$ we have $|xy| = |x||y|$.

(3) $|x + y| \leq |x| + |y| \;\; \forall \;\; x, y \in K$.

If instead of (3) the absolute value satisfies the stronger condition

(4) $|x + y| \leq max(|x|, |y|)$

then we say that it is a **valuation** or that the absolute value is non-archimedean. A subring D of K is called a **valuation ring** if it has the property that for any $x \in K$ we have $x \in D$ or $x^{-1} \in D$. The valuation rings give rise to valuations and valuations give valuation rings which is shown as follows.

Let D be a valuation ring of K and let $\mathbf{u}$ be the group of units of D. We claim that D is a local ring. Indeed, suppose $x, y \in D$ are not units and $\frac{x}{y} \in D$. Then

$$1 + \frac{x}{y} = \frac{x+y}{y} \in D.$$

If $x + y$ were a unit then $y^{-1} \in D$, contradicting the assumption that y is not a unit. We see that for $z \in D, zx$ is not a unit. Hence the non-units of D form an ideal which must therefore be the unique maximal ideal of D.

Now, let $\mathbf{m}$ be the maximal ideal of D and let $\mathbf{m}^*$ be the multiplicative system of nonzero elements of $\mathbf{m}$. Then

$$K^* = \mathbf{m}^* \cup \mathbf{u} \cup (\mathbf{m}^*)^{-1}$$

is disjoint union of $\mathbf{m}^*, \mathbf{u}$ and $(\mathbf{m}^*)^{-1}$. Factor group $K^*/\mathbf{u}$ can now be given an ordering. Let $x \in K^*$ and let us denote the coset $x\mathbf{u}$ of the factor group $K^*/\mathbf{u}$ by $|x|$.
Define $|x| < 1$ iff $x \in \mathbf{m}^*$.
Note: if $x, y \in K$ and $x, y \neq 0$, then

$$|x| < |y| \Longleftrightarrow |x/y| < 1 \Longleftrightarrow x/y \in \mathbf{m}^*$$

Conversely, given a valuation of K we let D be the subset of K consisting of all x such that $|x| \leq 1$. Then by axioms of valuation, D is a ring. If $|x| < 1$ then $|x^{-1}| > 1$ so that x^{-1} is not in D. If $|x| = 1$ then $|x^{-1}| = 1$. Thus D is a valuation ring whose maximal ideal consists of those elements x with $|x| < 1$ and whose units consist of those elements x with $|x| = 1$.
Definition 5: A **discrete valuation** on a field K is a function $\nu : K^\times \to \mathbf{Z}$ satisfying
(1) ν is surjective,
(2) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K^\times$
(3) $\nu(x + y) \geq min\{\nu(x), \nu(y)\}$ for all $x, y \in K^\times$ with $x + y \neq 0$
The subring $\{x \in K : \nu(x) \geq 0\} \cup \{0\}$ is called the **valuation ring** of ν.
Definition 6: An integral domain R is called a **Discrete Valuation Ring** (D.V.R.) if R is the valuation ring of a discrete valuation ν on the field of fractions of R. In other words a discrete valuation ring is a principal ideal ring having a unique prime ideal. It is therefore a local ring.
Example: The localization $\mathbf{Z}_{(p)}$ of $\mathbf{Z}$ at any nonzero prime ideal (p) is a D.V.R. with respect to the discrete valuation ν_p on $\mathbf{Q}$ defined as follows. Every element $a/b \in \mathbf{Q}^\times$ can be written uniquely in the form

$p^n(a_1/b_1)$ where $n \in \mathbf{Z}, a_1/b_1 \in \mathbf{Q}^\times$ and both a_1 and b_1 are relatively prime to p. We know that the $p-$ *adic valuation* on $\mathbf{Q}$ is defined as

$$\nu_p(\frac{a}{b}) = \nu_p(p^n \frac{a}{b}) = n.$$

We can easily check that the axioms for a D.V.R. are satisfied. The corresponding valuation ring is the set of rational numbers with $n \geq 0$ together with 0, i.e. the rational numbers a/b where b is not divisible by p, which is $\mathbf{Z}_p$.

References

(1) Abstract Algebra, David S. Dummit and Richard M. Foote, Prentice Hall, Second Edition.
(2) Algebra, Serge Lang, Springer, Revised Third Edition.
(3) Number Theory, Serge Lang, Springer.