INVERSE LIMITS, CATEGORIES, AND FUNCTORS (MAT 522)

Let $\{G_n\}$ be a sequence of groups for $n \geq 0$. Further assume for all $n \geq 1$ there exist surjective homomorphisms $f_n : G_n \to G_{n-1}$.

Next, form infinite sequences $x = (x_0, x_1, x_2, ...)$ such that $x_{n-1} = f_n(x_n)$.

Since we have surjectivity, we can take any $x_n \in G_n$ to G_{n+1} utilizing f_{n+1}. So, these infinite sequences exist, projecting to any given x_0. Define multiplication of these sequences componentwise, and it is clear that the set of such sequences form a group. We call this group the **i**_nverse_ **l**_imit_ of the family $\{(G_n, f_n)\}$. We denote the inverse limit by $\overset{\lim}{\leftarrow} (G_n, f_n)$, or simply $\lim G_n$ if the reference to f_n is clear.

Let us give an example that deals with a subject familiar to us from the beginning of the course. Example: Let $G_n = \mathbb{Z}/p^{n+1}\mathbb{Z}$ for each $n \geq 0$. Let $f_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ be the canonical homomorphism. Then f_n is surjective, and the limit is called the group of $p-$ **adic integers**, denoted by $\mathbb{Z}_p$.

A **category** $\mathcal{A}$ consists of a collection of objects $Ob(\mathcal{A})$; and for three object $A, B, C \in Ob(\mathcal{A})$ a law of composition holds (i.e. a map) $Mor(B, C) \times Mor(A, B) \to Mor(A, C)$ satisfying the following axioms:

CAT 1. Two sets $Mor(A, B)$ and $Mor(A', B')$ are disjoint unless $A = A'$ and $B = B'$, in which case they are equal.

CAT 2. For each object A of $\mathcal{A}$ there is a morphism $id_A \in Mor(A, A)$ which acts as left and right identity for the elements of $Mor(A, B)$ and $Mor(B, A)$ respectively, for all objects $B \in Ob(\mathcal{A})$.

CAT 3. The law of composition is associative (when defined), i.e. given $f \in Mor(A, B)$,$g \in Mor(B, C)$ and $h \in Mor(C, D)$ then $(h \circ g) \circ f = h \circ (g \circ f)$, for all objects A, B, C, D of $\mathcal{A}$.

A morphism f is called an **isomorphism** if there exists a morphism $g : B \to A$ such that $g \circ f$ and $f \circ g$ are the identities in $Mor(A, A)$ and $Mor(B, B)$ respectively. If $A = B$, then we also say that the isomorphism is an **automorphism**.

A morphism of an object A into itself is called an **endomorphism**.

The set of endomorphisms of A is denoted by $End(A)$.
Also, $Aut(A)$ denotes the set of automorphisms of A.
A simple example is **Grp** as the category of groups, or the category whose objects a groups and whose morphisms are group-homomorphisms. We can see that the three axioms are trivially satisfied.

Let $\mathcal{A}$, $\mathcal{B}$ be categories. A **covariant functor** F of $\mathcal{A}$ into $\mathcal{B}$ is a rule which to each object A in $\mathcal{A}$ associates an object $F(A)$ in $\mathcal{B}$, and to each morphism $f : A \to B$ associates a morphism $F(f) : F(A) \to F(B)$ such that:
FUN 1. For all A in $\mathcal{A}$ we have $F(id_A) = id_{F(A)}$.
FUN 2. If $f : A \to B$ and $g : B \to C$ are two morphisms of $\mathcal{A}$ then $F(g \circ f) = F(g) \circ F(f)$.

A quick example that is relatively easy to see is as follows:
Associate each group G to its set (stripped of the group structure) we obtain a functor from the category of groups into the category of sets, provided that we associate with each group-homomorphism itself, viewed only as a set-theoretic map. Such a functor is called a **stripping** functor or **forgetful functor**.

We observe that a functor transforms isomorphisms into isomorphisms, because $f \circ g = id$ implies $F(f) \circ F(g) = id$ also.
We can define the notion of a **contravariantfunctor** from $\mathcal{A}$ into $\mathcal{B}$ by using essentially the same definition, but reversing all arrows $F(f)$, i.e. to each morphism $f : A \to B$ the contravariant functor associates a morphism $F(f) : F(B) \to F(A)$ (going in the opposite direction), such that, if $f : A \to B$ and $g : B \to C$ are morphisms in $\mathcal{A}$, then $F(g \circ f) = F(f) \circ F(g)$.
Sometimes a functor is denoted by writing f_* instead of $F(f)$ in the case of a covariant functor, and by writing f^* in the case of a contravariant functor.

Another example more closely related to class. Let us test the functor of parabolic induction, $i_{G,M}$ to make sure it is a covariant functor. In this case, the categories are $AlgM$ and $AlgG$ with the morphisms being all intertwining operators $\varphi : V_1 \to V_2$. So, this carries all smooth represenations $(\sigma, V) \in AlgM$ to smooth representations $(i_{G,M}(\sigma), i_{G,M}(V)) \in AlgG$, and associates each intertwining operator φ for two smooth representations V_1, V_2 in $AlgM$ to the intertwining operator $i_{G,M}(\varphi)$ for the two smooth representations $i_{G,M}(V_1), i_{G,M}(V_2)$ in $AlgG$. Let us check the axioms, **FUN 1**., and **FUN 2**..

For all V in $AlgM$, we have $i_{G,M}(id_V) = id_{i_{G,M}(V)}$. Basically, the identity intertwining operator from one representation V of $AlgM$ carried under the map of $i_{G,M}$ is equal to the identity intertwining operator for $i_{G,M}(V)$ with the id function held stable. This is clearly the case as their is only one identity intertwining operator for each $V \in AlgM$ and for each $i_{G,M}(V)$, as can be seen by Schur's Lemma with scalar $\lambda = 1$.
If $\varphi_1 : V_1 \to V_2$ and $\varphi_2 : V_2 \to V_3$ are two morphisms of $AlgM$, then $i_{G,M}(\varphi_2 \circ \varphi_1) = i_{G,M}(\varphi_2) \circ i_{G,M}(\varphi_1)$. So, if we have two intertwining operator and compose them, then carry them under $i_{G,M}$ we get the intertwining operator from $i_{G,M}(V_1)$ to $i_{G,M}(V_3)$. This is clearly the same as composing the interwtining operator from $i_{G,M}(V_1)$ to $i_{G,M}(V_2)$ with the intertwining operator from $i_{G,M}(V_2)$ to $i_{G,M}(V_3)$.
Similary it is easy to see that $r_{G,M}$ is a covariant functor. The contragredient is a contravariant functor with all the arrows simply reversing.

References

[1] Serge Lang. *Inverse limit and completion*, Algebra. **3rd edition** (1993), 49-53, 161-169, 313-328.
[2] Serge Lang. *Categories and functors*, Algebra. **3rd edition** (1993), 53-66.