Measures of Departure from Constant Failure Rate Models and Proportional Hazards Rate Models for Grouped Data

Bhaskar Bhattacharya

Department of Mathematics, Southern Illinois University

Carbondale, IL 62901-4408, USA

Abstract

Two measures are proposed to represent the degree of departure from the constant failure rate model of a system when data are grouped. Two measures are also proposed to represent the degree of departure from the proportional hazards rate model when two systems are present and grouped data are considered. In each case one measure is based on the Kullback-Leibler discrepancy and the other is based on the Pearson chi-square type discrepancy using the failure rates. The usefulness of the proposed measures are discussed with applications. A simulation study shows that the proposed measures perform no worse than the goodness-of-fit tests when testing for the constant failure rate model.

Key words and phrases: Delta method, goodness-of-fit tests, Kullback-Leibler information, Mantel-Haenszel test, Pearson chi-square type discrepancy.

1 Introduction

Although the length of life of a system is a continuous random variable, the observations are often collected as grouped data due to limitations of monitoring devices. For example, if a system is monitored periodically, length of life can be defined as the maximum number of time periods successfully completed, or if a machine operates in cycles, length of life refers to the number of cycles successfully completed prior to the failure. Thus continuous type data are often collected at finite number of specified time points. Hence we assume that the time to failure T is discrete with k possible values labeled $t_1, t_2, \ldots, t_k$. We assume that all failures are observed; thus t_k could be large. For $1 \leq i \leq k$, let p_i denote the probability that an object chosen at random will fail at time t_i, where $p_i \geq 0$, $\sum_{i=1}^{k} p_i = 1$. Let θ_i be the conditional probability that an object will fail at time t_i given that it has not failed before; thus θ_i is the discrete failure rate at t_i and can be expressed as $\theta_i = p_i / \sum_{j=i}^{k} p_j$, $1 \leq i \leq k-1$. Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$.

We have considered the Kullback-Leibler discrimination information and the Pearson chi-square discrepancy measures to detect deviations of failure rates from the constant failure rate (CF) model and the proportional hazard rate (PH) model. Both of these measures have been proved to be useful in literature in a variety of situations (see Read and Cressie, 1988, Tomizawa, 1991, 1994, and the references therein). In general, testing for CF model is equivalent to testing for exponentiality. We refer the reader to Ebrahimi et al. (1992) for a test procedure and related references. Tests for and against the PH model for a grouped data are given in Lawless (1982). Model diagnostic techniques are also used for this purpose (see Lawless, 1982, Ch. 6, 7 for details and other references). Aranda-Ordaz (1980) attempted to model departures from the PH model. Schoenfeld (1980) proposed a chi-square test when testing for PH model which is not too closely tied to parametric assumptions; however according to Lawless (1982) this test is rather complicated. Ebrahimi and Kirmani (1996 a,b) proposed a measure of discrepancy between two general residual-life distributions based on Kullback-Leibler discrimination information taking account of the current age of the system. Clearly, there is interest for measures to detect violations from CF and PH models.

In this paper we adopt a nonparametric approach. In Section 2, we propose two measures which detect deviations of the failure rates of a system from the CF model. In Section 3, we propose two separate measures which detect deviations of the failure rates of two systems from the PH model. We have used confidence intervals to detect the presence of CF or PH model. Assuming the null hypothesis of CF or PH model is not true, the present treatment provides an easy alternative to testing for the CF and PH models in respective cases. In Section 4, we have used data sets to demonstrate the usefulness of the proposed measures. We have conducted a simulation study for the CF case for comparison between the proposed measures, the likelihood ratio (G^2) and the Pearson chi-square (X^2) tests. We conclude with some final remarks and discussions in Section 5.

2 Measures of Departure from CF Model

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_k)$ be two probability vectors (PV). The Kullback-Leibler information discrepancy and the Pearson chi-square discrepancy between $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined as

$$I(\boldsymbol{x}|\boldsymbol{y}) = \sum_{j=1}^{k} x_j \ln \frac{x_j}{y_j} \qquad \text{and} \qquad P(\boldsymbol{x}|\boldsymbol{y}) = \sum_{j=1}^{k} \frac{(x_j - y_j)^2}{y_j},$$

where $x_i \geq 0$, $y_i > 0$ and $0 \ln 0 = 0$ by convention. Neither $I(\boldsymbol{x}|\boldsymbol{y})$ nor $P(\boldsymbol{x}|\boldsymbol{y})$ is a distance measure, although it can be verified that each is nonnegative and is equal to zero if and only if $\boldsymbol{x} = \boldsymbol{y}$ (Read and Cressie, 1988).

Let $v_j = \theta_j / \sum_{i=1}^{k-1} \theta_i$, $j = 1, \ldots, k-1$ and $v_0 = (k-1)^{-1}$. Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_{k-1})$ and $\boldsymbol{v}_0$ be the vector of length $k-1$ with each coordinate being equal to v_0. Consider

$$\phi_1 = \frac{1}{\ln\,(k-1)} I(\boldsymbol{v}|\boldsymbol{v}_0) \quad \text{and} \quad \psi_1 = \frac{1}{k-2} P(\boldsymbol{v}|\boldsymbol{v}_0).$$

It follows using algebra that

$$I(\boldsymbol{v}|\boldsymbol{v}_0) = \ln\,(k-1) - H(\boldsymbol{v})$$

where $H(\boldsymbol{v}) = -\sum_{j=1}^{k-1} v_j \ln v_j$ is the entropy of $\boldsymbol{v}$. Since $0 \leq H(\boldsymbol{v}) \leq \ln\,(k-1)$, it follows that $0 \leq \phi_1 \leq 1$. Also,

$$P(\boldsymbol{v}|\boldsymbol{v}_0) = (k-1) \sum_{i=1}^{k-1} v_j^2 - 1 \leq k-2$$

from which it follows that $0 \leq \psi_1 \leq 1$. Clearly, a system has CF model if and only if $\phi_1 = \psi_1 = 0$ and the degree of departure from constant failure rate is largest if and only if $\phi_1 = \psi_1 = 1$ which happens if and only if all the failures occur at one single time point. Since $I(\boldsymbol{v}|\boldsymbol{v}_0)$ and $P(\boldsymbol{v}|\boldsymbol{v}_0)$ are divergence measures which provide a measure of 'distance' from uniformity, a higher value of divergence means farther 'distance' from uniformity. Thus the degree of departure from the CF model increases if and only if the value of $\phi_1,\ \psi_1$ increases.

Let $n = \sum_{i=1}^{k} n_i$ items are put on test, where n_i denote the observed frequency for the ith time point. Assuming that the $\{n_i\}$ result from a full multinomial sampling, the sample versions of the two discrepancies, $\hat{\phi}_1,\ \hat{\psi}_1$ are constructed by estimating p_i with n_i/n in the expressions of $\phi_1,\ \psi_1$ respectively. We use the delta method (Serfling, 1980) to obtain the asymptotic normal distributions (with means and variances) for $\hat{\phi}_1$ and $\hat{\psi}_1$. Using the approximate standard errors, large sample confidence intervals are constructed.

Assume that the CF model is not true. Let $\hat{\theta}_i = \hat{p}_i / \sum_{j=i}^{k} \hat{p}_j$, $1 \leq i \leq k-1$ and $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{k-1})$. It can be shown that $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{\mathcal{L}} MVN(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})$ where $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is a diagonal matrix with jth diagonal entry is $\theta_j(1-\theta_j)/P_j$ with $P_j = \sum_{i=j}^{k} p_i$. Using this result it follows that $\sqrt{n}(\hat{\phi}_1 - \phi_1)$ and $\sqrt{n}(\hat{\psi}_1 - \psi_1)$ have asymptotically normal distributions with means 0 and variances $\sigma^2(\phi_1)$ and $\sigma^2(\psi_1)$, respectively, where

$$\sigma^2(\phi_1) = \frac{1}{(\ln\,(k-1))^2 T^2} \sum_{j=1}^{k-1} \frac{\theta_j(1-\theta_j)}{P_j} \left(\ln\,\theta_j - \frac{A}{T}\right)^2$$

and

$$\sigma^2(\psi_1) = \frac{4}{(k-2)^2 T^4} \sum_{j=1}^{k-1} \frac{\theta_j(1-\theta_j)}{P_j} \left(\theta_j - \frac{B}{T}\right)^2$$

and $T = \sum_{i=1}^{k-1} \theta_i,\ A = \sum_{i=1}^{k-1} \theta_i \ln\,\theta_i,\ B = \sum_{i=1}^{k-1} \theta_i^2$.

Let $\hat{\sigma}(\phi_1)$ be the estimate of $\sigma(\phi_1)$ obtained by estimating p_i with $\hat{p}_i$. Then $\hat{\sigma}(\phi_1)/\sqrt{n}$ is an estimated standard error for $\hat{\phi}_1$ and $\hat{\phi}_1 \pm z_{\alpha/2}\hat{\sigma}(\phi_1)/\sqrt{n}$ is an approximate $100(1-\alpha)\%$ confidence interval for ϕ_1 where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point from the normal distribution. A $100(1-\alpha)\%$ confidence interval for ψ_1 can be constructed in a similar way.

3 Measures of Departure from PH Model

When comparing between two systems, let $n = \sum_{i=1}^{k} n_i$ items from system 1 and $m = \sum_{i=1}^{k} m_i$ items from system 2 are put on test, where n_i (m_i) denote the observed failure frequency for the ith time point for system 1 (2). We assume that both the $\{n_i\}$ and $\{m_i\}$ result from full multinomial sampling. We assume that the time to failure T is discrete with k possible values labeled $t_1, t_2, \ldots, t_k$. For $1 \le i \le k$, let p_i (q_i) denote the probability that an item from system 1 (2) chosen at random will fail at time t_i, where $p_i > 0$, $\sum_{i=1}^{k} p_i = 1$ and $q_i > 0$, $\sum_{i=1}^{k} q_i = 1$. Let θ_i (η_i) be the discrete failure rate at time t_i for system 1 (2) and can be expressed as $\theta_i = p_i / \sum_{j=i}^{k} p_j$ ($\eta_i = q_i / \sum_{j=i}^{k} q_j$), $1 \le i \le k-1$. Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_{k-1})$. For the failure rates θ_j's and η_j's, define the corresponding PV's $\boldsymbol{v} = (v_1, v_2, \ldots,\ v_{k-1})$

and $\boldsymbol{u} = (u_1, u_2, \ldots, u_{k-1})$ with ith coordinates as $v_i = \theta_i / \sum_{j=1}^{k-1} \theta_j,\ u_i = \eta_i / \sum_{j=1}^{k-1} \eta_j$ respectively, for $1 \leq i \leq k-1$. Then it is easy to see that $v_i = u_i$ if and only if $\theta_i = c\eta_i$, for all i for a constant c. Thus $\boldsymbol{v} = \boldsymbol{u}$ if and only if $\{\theta_i\}$ and $\{\eta_i\}$ are proportional hazards.

Consider the divergence measure given by $I(\boldsymbol{v}|\boldsymbol{u})$ which can be expressed as

$$\begin{aligned} I(\boldsymbol{v}|\boldsymbol{u}) &= \textstyle\sum_{j=1}^{k-1} v_j \ln \frac{v_j}{u_j} \\ &= \frac{M}{T} + \ln \frac{N}{T} \end{aligned}$$

where $M = \sum_{i=1}^{k-1} \theta_i \ln (\theta_i/\eta_i),\ N = \sum_{i=1}^{k-1} \eta_i$. Consider the divergence measure given by $P(\boldsymbol{v}|\boldsymbol{u})$ which can be expressed as

$$\begin{aligned} P(\boldsymbol{v}|\boldsymbol{u}) &= \textstyle\sum_{j=1}^{k-1} \frac{(v_j - u_j)^2}{u_j} \\ &= \textstyle\sum_{j=1}^{k-1} \frac{v_j^2}{u_j} - 1 \\ &= \frac{NC}{T^2} - 1 \end{aligned}$$

where $C = \sum_{j=1}^{k-1} \theta_j^2 / \eta_j$.

To obtain measures between 0 and 1, we consider the transformations

$$\phi_2 = \sqrt{\frac{I(\boldsymbol{v}|\boldsymbol{u})}{1 + I(\boldsymbol{v}|\boldsymbol{u})}} \quad \text{and} \quad \psi_2 = \sqrt{\frac{P(\boldsymbol{v}|\boldsymbol{u})}{1 + P(\boldsymbol{v}|\boldsymbol{u})}}.$$

Clearly, the value of $\phi_2 = \psi_2 = 0$ correspond to the case of proportional hazard rates. Parallel to the discussion in the CF case, the value of $\phi_2,\ \psi_2$ increases if and only if the divergence of the hazard rates from being proportional hazards also increases.

The sample versions of the discrepancies $\hat{\phi}_2,\ \hat{\psi}_2$ are constructed by estimating p_i (q_i) with n_i/n (m_i/m) in the expressions of $\phi_2,\ \psi_2$, respectively. Using the delta method, the asymptotic normal distributions of $\hat{\phi}_2,\ \hat{\psi}_2$ are found, and consequently the asymptotic confidence intervals can be constructed.

Assume that the PH model is not valid. Let $\hat{\theta}_i = \hat{p}_i / \sum_{j=i}^{k} \hat{p}_j,\ \hat{\eta}_i = \hat{q}_i / \sum_{j=i}^{k} \hat{q}_j,\ 1 \leq i \leq k-1$, $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{k-1})$ and $\hat{\boldsymbol{\eta}} = (\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_{k-1})$. First we assume that both

$m, n \to \infty$ in such a way that $n/m \to \lambda$, for some $0 < \lambda < \infty$. Then by the delta method, it follows

$$\sqrt{n+m}\left(\begin{pmatrix} \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\eta}} \end{pmatrix} - \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\eta} \end{pmatrix}\right) \xrightarrow{\mathcal{L}} MVN\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} (1+\lambda^{-1})\boldsymbol{\Sigma}_{\boldsymbol{\theta}} & \mathbf{0} \\ \mathbf{0} & (1+\lambda)\boldsymbol{\Sigma}_{\boldsymbol{\eta}} \end{pmatrix}\right)$$

where $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is defined earlier and $\boldsymbol{\Sigma}_{\boldsymbol{\eta}}$ is a diagonal matrix with jth diagonal entry is $\eta_j(1-\eta_j)/Q_j$ where $Q_j = \sum_{i=j}^k q_i$. Hence it follows that $\sqrt{n+m}(\hat{\phi}_2 - \phi_2)$ and $\sqrt{n+m}(\hat{\psi}_2 - \psi_2)$ have asymptotically normal distributions with means 0 and variances $\sigma^2(\phi_2)/4\phi_2(\phi_2+1)^3$ and $\sigma^2(\psi_2)/4\psi_2(\psi_2+1)^3$, respectively, where

$$\begin{aligned} \sigma^2(\phi_2) &= \tfrac{1}{T^2}\Big\{(1+\lambda^{-1})\textstyle\sum_{j=1}^k \frac{\theta_j(1-\theta_j)}{P_j}\Big[\ln\frac{\theta_j}{\eta_j} - \frac{M}{T}\Big]^2 \\ &\qquad + (1+\lambda)\textstyle\sum_{j=1}^k \frac{\eta_j(1-\eta_j)}{Q_j}\Big[\frac{\theta_j}{\eta_j} - \frac{T}{N}\Big]^2\Big\} \end{aligned}$$

and

$$\begin{aligned} \sigma^2(\psi_2) &= \tfrac{N^2}{T^4}\Big\{4(1+\lambda^{-1})\textstyle\sum_{j=1}^k \frac{\theta_j(1-\theta_j)}{P_j}\Big[\frac{\theta_j}{\eta_j} - \frac{C}{T}\Big]^2 \\ &\qquad + (1+\lambda)\textstyle\sum_{j=1}^k \frac{\eta_j(1-\eta_j)}{Q_j}\Big[\frac{\theta_j^2}{\eta_j^2} - \frac{C}{N}\Big]^2\Big\}. \end{aligned}$$

Let $\hat{\sigma}(\phi_2)$ be the estimate of $\sigma(\phi_2)$ obtained by estimating p_i (q_i) with $\hat{p}_i$ $(\hat{q}_i)$. Also we estimate λ by $\hat{\lambda} = n/m$. Then $\hat{\sigma}(\phi_2)/\sqrt{4(n+m)\hat{\phi}_2(\hat{\phi}_2+1)^3}$ is an estimated standard error for $\hat{\phi}_2$ and $\hat{\phi}_2 \pm z_{\alpha/2}\hat{\sigma}(\phi_2)/\sqrt{4(n+m)\hat{\phi}_2(\hat{\phi}_2+1)^3}$ is an approximate $100(1-\alpha)\%$ confidence interval for ϕ_2. A $100(1-\alpha)\%$ confidence interval for ψ_2 can be constructed in a similar way.

4 Examples and Simulation

First we consider a data, from Cox (1959) (also in Lawless, 1982, p. 505) originally analyzed by Mendenhall and Hader (1958). These data (Data 1) are on failure times for 369 radio transmission receivers. These failures are classified as confirmed on arrival at the maintenance center (type I) or unconfirmed (type II). It is of interest to test the null hypothesis that the hazard rates are proportional, that is, the type of failure and time to failure are independent. Like Cox (1959), we assume that the two risks are independent. Forty four of the 369 receivers did not fail during the test period (630 hours). Since, these censored items give no information about the failure rates of the two systems, we have excluded them from consideration.

We also consider a second data set (Data 2) from Lawless (1982, p.257) which reports the number of cycles to failure for a group of 60 electrical appliances in a life test.

******** INSERT TABLE 1 ABOUT HERE ********

Table 1 gives the estimates of the proposed dispersion measures for these data, the corresponding 95% confidence intervals and the X^2, G^2 values and their p-values. When considering the first data set separately for type I and total, the confidence intervals based on $\hat{\phi}_1$ and $\hat{\psi}_1$ do not contain the value zero, and hence the hypothesis of CF model cannot be assumed. This conclusion is supported by the goodness-of-fit statistics (corresponding X^2, G^2 values, chi-square with 12 degrees of freedom). For type II data the confidence interval based on $\hat{\phi}_1$ does contain zero, so the hypothesis of CF model can be assumed which is also supported by the goodness-of-fit statistics. However for type II data, the confidence interval based on $\hat{\psi}_1$ does not contain zero, so the hypothesis of CF model cannot be assumed when $\hat{\psi}_1$ is used.

For the second data set, none of the intervals contains the value zero. So one would conclude that the CF model does not hold when the dispersion measures $\hat{\phi}_1$, $\hat{\psi}_1$ are used. Also from the corresponding X^2, G^2 values (chi-square with 8 degrees of freedom), the hypothesis of CF model is clearly rejected.

Using the first data set with measures $\hat{\phi}_2$, $\hat{\psi}_2$, none of the intervals contains zero so that the hypothesis of PH model cannot be assumed when these measures are used.

When testing the PH model as a null hypothesis, the values of the X^2, G^2 (chi-square with 12 degrees of freedom) indicate that in both cases the PH model is accepted.

Often a logistic model is useful when analyzing grouped data. Under this model, a Mantel-Haenszel test considers testing equality of the survival functions against the PH model (see Lawless, 1982, p.383). The value of this test statistic is 4.32 (p-value=0.0377) which has an asymptotic chi-square distribution with 1 degree of freedom. Thus the PH model is accepted using this test also.

To compare the effectiveness of the proposed measures with the standard chi-square tests and to investigate their small sample behavior a modest simulation study is performed. We consider the case of departures from the CF model using the measures $\hat{\phi}_1$, $\hat{\psi}_1$ and the standard chi-square goodness-of-fit tests. Since the approximate normal distribution of the proposed measures is only valid when the CF model is not true and that the comparisons with the chi-square tests are done on equal footing, we consider a hypothesis testing procedure as follows: let $H_0 : \phi_1 = \phi_0 \neq 0$ versus $H_1 : \phi_1 > \phi_0$ and reject H_0 if the test statistic $\sqrt{n}(\hat{\phi}_1 - \phi_0)/\hat{\sigma}(\phi_1)$ is greater than z_α. Similarly for the case of $\hat{\psi}_1$. We have used the value of $\phi_0 = \psi_0 = .0001$ which is close enough to 0 corresponding to the CF model. We have chosen the θ_i's to represent deviation from the CF model in a variety of ways, and then the p_i's are obtained by using the transformation $p_i = \theta_i \prod_{j=1}^{i-1}(1 - \theta_j)$, $1 \leq i \leq k-1$, $p_k = \prod_{j=1}^{k-1}(1 - \theta_j)$. Samples are taken from multinomial distribution with these probabilities and four above tests are performed. We consider samples of sizes 30, 60, 100 and 200.

The likelihood ratio test (G^2) and the chi-square (X^2) tests are given by

$$G^2 = \sum_{i-1}^{k} n_i \ln \frac{n_i}{e_i} \quad \text{and} \quad X^2 = \sum_{i=1}^{k} \frac{(n_i - e_i)^2}{e_i}$$

where $e_i = n\hat{p}_i^0$ are the expected values, $\hat{p}_i^0$ are obtained using the above transformations by replacing each θ_j by $\sum_{i=1}^{k-1} \hat{p}_i / \sum_{i=1}^{k-1} \sum_{j=i}^{k} \hat{p}_j$. We have considered $k = 5$

and $\alpha = .05$ with 10,000 repetitions in each case. The estimated powers are given in Table 2.

******** INSERT TABLE 2 ABOUT HERE ********

From this simulation study, $\hat{\psi}_1$ has largest estimated powers at almost all the alternatives chosen. When sample size is small, $\hat{\psi}_1$ still performs best whereas $\hat{\phi}_1$ does not do as well especially near the CF model. When close to the CF model, the standard chi-square tests perform poorly even when the sample size is as large as 200. Both of $\hat{\phi}_1$, $\hat{\psi}_1$ performed competitively or better than the standard chi-square tests in this situation. When one θ_i value is much larger than all others, this situation is farthest from the CF model and all the tests detected this situation quite effectively. A similar effectiveness is observed when the failure rates are fluctuating rapidly. In general, we have observed that the performance of all four tests depend not only on the spacing of θ_i's but also on their actual values. Similar behavior pattern is observed at other θ_i values which are not reported here for brevity.

We have noted that standard error of $\hat{\phi}_1$ is larger than the standard error of $\hat{\psi}_1$ resulting in wider confidence intervals for $\hat{\phi}_1$. This pattern may also be noted in the examples worked out.

5 Discussion

We have considered two types of measures for detecting deviations from CF and PH models for different systems. These measures would be useful for comparing the degree of departure from the respective models among different data sets with same number of groups. These measures are nonparametric and are based on failure rates directly, and hence this is a more natural procedure for comparison among failure rates than the tests discussed earlier. Since, the CF (PH) model describes equality (proportionality) of the failure rates it seems natural to use divergence measures to describe the degree of nonuniformity among the failure rates. Thus the measures proposed are preferable to $G^2/n,\ X^2/n$ when one wants to see with a single summary measure how far *the failure rates are distant from uniformity or proportionality.* On the other hand, $G^2/n,\ X^2/n$ would be preferable when one wants to compare between the observed and the expected cell frequencies under the CF or PH model.

In $G^2/n,\ X^2/n$ tests, different time points are weighted according to the numbers still at risk. However, for the proposed measures all the time points are equally weighted. From the simulation study, it appears that the proposed measures perform no worse than the standard chi-square tests near CF model for small sample sizes.

The measures proposed are not influenced when each frequency is multiplied or divided by a constant but of course their standard errors are. Some confidence intervals in Table 1 include negative values of the measures which are impossible. So very large sample sizes would be needed for the delta method to work well in these cases.

Acknowledgements

The author is grateful to the referees for valuable comments which have improved this paper.

References

Aranda-Ordaz, F. J. (1980). Extensions of the proportional hazards model. Paper presented at the 1980 ASA Meetings, Houston, Texas, August 11-14, 1980.

Cox, D. R. (1959). The analysis of exponentially distributed lifetimes with two types of failures. *Journal of Royal Statistical Society–Series B* **21**, 411-421.

Ebrahimi, N., Habibullah, M. and Soofi, E. S. (1992). Testing exponentiality based on Kullback-Leibler information. *Journal of Royal Statistical Society–Series B* **54**, 739-748.

Ebrahimi, N. and Kirmani, S. N. U. A. (1996). A measure of discrimination between two residual life-time distributions and its applications. *Annals of the Institute of Statistical Mathematics* **48**, 257-265.

Ebrahimi, N. and Kirmani, S. N. U. A. (1996). A characterization of the proportional hazards model through a measure of discrimination between two residual life distributions, *Biometrika* **83**, 233-235.

Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons, New York.

Mendenhall, W. and Hader, R. J. (1958). Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data. *Biometrika* **45**, 504-520.

Read, T. R. C. and Cressie, N. (1988). *Goodness-of-fit Statistics for Discrete Multivariate Data*. Springer-Verlag, New York.

Schoenfeld, D. (1980). Chi-squared goodness-of-fit tests for the proportional hazards regression model. *Biometrika* **67**, 145-153.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics.* John Wiley & Sons, New York.

Tomizawa, S. (1991). Shannon entropy type measure of departure from uniform association in cross-classifications having ordered categories. *Statistics & Probability Letters* **11**, 547-550.

Tomizawa, S. (1994). Two kinds of measures of departure from symmetry in square contingency tables having nominal categories. *Statistica Sinica* **4**, 325-334.

Table 1: Dispersion measures, their confidence intervals and the corresponding goodness-of-fit test statistics for two examples

Data	Estimated measure	Standard error	95% Confidence interval	X^2 test (p-value)	G^2 test (p-value)
Data 1, type I	$\hat{\phi}_1 = 0.0395$	0.0148	(0.0104, 0.0686)	33.628	33.451
	$\hat{\psi}_1 = 0.0215$	0.0008	(0.0199, 0.0231)	(0.0008)	(0.0008)
Data 1, type II	$\hat{\phi}_1 = 0.0364$	0.0242	(−0.0110, 0.0839)	14.948	16.289
	$\hat{\psi}_1 = 0.0188$	0.0013	(0.0163, 0.0212)	(0.2443)	(0.1783)
Data 1, total	$\hat{\phi}_1 = 0.0340$	0.0120	(0.0106, 0.0574)	39.766	42.241
	$\hat{\psi}_1 = 0.0184$	0.0006	(0.0171, 0.0196)	(0.0001)	(0.0000)
Data 2	$\hat{\phi}_1 = 0.1596$	0.0378	(0.0855, 0.2336)	59.526	43.796
	$\hat{\psi}_1 = 0.1155$	0.0040	(0.1077, 0.1233)	(0.0000)	(0.0000)
Data 1	$\hat{\phi}_2 = 0.1884$	0.0355	(0.1189, 0.2579)	9.47	4.96
	$\hat{\psi}_2 = 0.2598$	0.0553	(0.1514, 0.3682)	(0.6623)	(0.9593)

Table 2: Estimated power of different tests

theta	Sample size	$\hat{\phi}_1$ test	$\hat{\psi}_1$ test	X^2 test	G^2 test
$\boldsymbol{\theta} = (.17, .19, .21, .23)$	30	0.082	0.899	0.052	0.050
	60	0.053	0.906	0.062	0.053
	100	0.057	0.914	0.081	0.067
	200	0.089	0.941	0.131	0.115
$\boldsymbol{\theta} = (.2, .25, .3, .35)$	30	0.125	0.937	0.124	0.097
	60	0.158	0.956	0.199	0.162
	100	0.254	0.977	0.317	0.279
	200	0.517	0.994	0.605	0.576
$\boldsymbol{\theta} = (.40, .55, .70, .85)$	30	0.647	0.902	0.233	0.279
	60	0.691	0.992	0.535	0.615
	100	0.853	1.000	0.833	0.865
	200	0.990	1.000	0.994	0.996
$\boldsymbol{\theta} = (.2, .4, .6, .8)$	30	0.856	0.998	0.814	0.807
	60	0.987	1.000	0.988	0.985
	100	1.000	1.000	1.000	1.000
$\boldsymbol{\theta} = (.2, .6, .3, .8)$	30	0.824	0.999	0.887	0.885
	60	0.989	1.000	0.996	0.995
	100	1.000	1.000	1.000	1.000
$\boldsymbol{\theta} = (.8, .1, .8, .1)$	30	0.627	0.664	0.846	0.860
	60	0.844	0.889	0.997	0.997
	100	0.953	0.976	1.000	1.000
	200	0.998	0.999	1.000	1.000
$\boldsymbol{\theta} = (.01, .01, .9, .01)$	30	0.952	0.952	1.000	1.000
	60	0.999	0.999	1.000	1.000