Testing Equality of Scale Parameters Against Restricted Alternatives for $m \geq 3$ Gamma Distributions with Unknown Common Shape Parameter

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Abstract

Shiue and Bain (1983) proposed an approximate F-test for the equality of the scale parameters of two gamma distributions with equal but unknown shape parameters. In this article, we propose a simple procedure to test equality of scale parameters of $m \geq 3$ gamma distributions against nonincreasing order. The test is based on Fisher's method of combining p-values. The actual size of the resulting test is investigated through Monte Carlo studies. Also asymptotic results are derived for the nominal test size. These can be used to obtain a test which achieves the desired size. The case of more general partial orders is discussed.

Key words and phrases: Approximate tests, Fisher's combination method, Monte Carlo studies, order restricted tests, partial orders.

1 Introduction

Over last few decades, the gamma distribution has arisen as one of the most important vehicles to model life-testing situations. Because of the flexibility in choice of the shape and scale parameters, a wide variety of lifetime data fits quite adequately to it. Among situations that lead to the gamma distribution are waiting time problems as it is well-known that the time to kth occurrence of a Poisson process follows a gamma distribution. In reliability studies and in life testing, the gamma distribution is used as a generalization of the exponential distribution which is also a popular choice for modeling purposes. The gamma distribution is suggested as the failure time model for a system under continuous maintenance, where the reliability may experience some growth or decay initially but then reaches a stable state as time goes on. The gamma distribution has also been used in weather analysis. In theoretical calculations, the gamma distribution arises as the sum of independent, identically distributed exponential random variables. Gamma distribution evolves in the testing of equality of variances of several independent normal distributions. Johnson and Kotz (1970) provides a good review of the gamma distribution including several applications in various fields.

The gamma distribution denoted by $G(x;\theta,\kappa)$, with θ and κ being the scale and the shape parameters respectively, has density as

$$\frac{1}{\Gamma(\kappa)\theta^{\kappa}}x^{\kappa-1}e^{-x/\theta}, \quad x>0, \quad \theta,\ \kappa>0. \tag{1}$$

The maximum likelihood estimation of κ can be approximated by the empirically determined formulas of Greenwood and Durand (1960) as given below

$$\begin{aligned}
\hat{\kappa} &= \tfrac{0.5000876+0.1648852S-0.0544274S^2}{S}, && 0<S\leq 0.5772\\
&= \tfrac{8.898919+9.059950S+0.9775373S^2}{S(17.79728+11.968477S+S^2)}, && 0.5772<S\leq 17\\
&= \tfrac{1}{S}, && 17<S
\end{aligned}$$

where $S = \ln{(A/G)}$, and A and G denote the arithmetic and geometric means respectively.

Inferences concerning the parameters of the gamma distribution are rather difficult mainly due to the fact that they are not of the conventional location-scale type. Bain and Engelhardt (1975) derived the exact tests of κ with θ being an unknown nuisance parameter. Engelhardt and Bain (1977) developed a conditional test for testing the scale parameter of a gamma distribution with unknown shape parameter. Grice and Bain (1980) proposed an approximate test for the mean of a gamma distribution with both parameters unknown. For the two-sample situation, Shiue and Bain (1983) proposed an approximate test for testing the equality of the scale parameters of two gamma distributions with unknown but common shape parameter. Also, Shiue *et al.* (1988) proposed an approximate test for testing the equality of means of two gamma distributions with unknown and unequal shape parameters.

Inference for parameters of more than two gamma distributions is quite rare in the existing literature. Recently, Tripathi *et al.* (1993) proposed a test for the parameters of $m \geq 2$ gamma distributions based on a generalized minimum chi-square procedure. Although this m-sample test is applicable in versatile testing situations with general unrestricted alternatives, it is asymptotic in nature, and for $m = 2$, the authors found the test by Shiue and Bain (1983) perform better in the equal but unknown shape parameter case. Mudholkar *et al.* (1993) considered a test of equality of variances of m normal distributions against restricted alternatives which essentially reduces to testing the equality of the scale parameters of m gamma distributions with shape parameters being a function of the sample sizes and hence known. Robertson *et al.* (1988) considered likelihood ratio tests for trend in the scale parameters when the shape parameters are known. Large sample approximations for the significance levels are obtained in terms of chi-bar squared distributions which are quite involved. For the simple order alternative and equality of the shape parameters case, approximations are provided in terms of special functions.

In this paper we develop a procedure for testing homogeneity of ordered scale parameters for $m \geq 3$ gamma distributions with a common shape parameter. The procedure is done in stages starting with two samples. For two samples we use the approximate test of Shiue and Bain (1983) to test the equality of the scale parameters. Assuming that these two scale parameters are equal, the samples are combined and then tested with a third sample by again following the two-sample test procedure. This procedure continues until all m samples are considered, and the p-values from $m-1$ steps are collected. Mudholkar and McDermott (1989) and McDermott and Mudholkar (1993) report several ways of combining p-values including Fisher's, Liptak's and Tippett's. As they found that Fisher's has better power characteristics, we use Fisher's method in our case.

In life-testing situations, for the gamma distribution the limiting hazard rate is $1/\theta$, so that testing equality of scale parameters is equivalent to testing equality of limiting hazard rates. Also under the assumption of equal shape parameters, the test of equal scale parameters is actually a test of the distributions being identical. Examples of some practical problems to which the proposed test can be applied are: suppose data on failure times (in minutes) for some electrical components from three or more brands in an accelerated life test are available, and we like to test equality of average failure times for these brands against nonincreasing order; to study the effect of radiation on rats, several rats (randomly assigned) are exposed to low, medium and high doses of radiation; the survival times (in weeks) are recorded; one likes to test equality of average survival times against nonincreasing order; and so on.

The rest of the paper is organized as follows. In Section 2, we describe the related results and the proposed test. In Section 3, we study the actual size of the proposed test. In Section 4, we derive the asymptotic results for the actual size of the test as

$\kappa \to \infty$ and as $\kappa \to 0$. In Section 5, the Monte Carlo methods are used for simulation of the actual size of the proposed test for $m = 3$ at different combinations of sample sizes and κ, and the results obtained are discussed. In Section 6, we discuss the situation of other partial orders. In Section 7, we make some concluding remarks.

2 The Proposed Test

For a random sample of size n from the gamma density (1), it follows that $2n\overline{X}/\theta \sim \chi^2_{2n\kappa}$ where χ^2_ν denotes a chi-square distribution with ν degrees of freedom. For the case of two independent gamma distributions with common shape parameters, let $\overline{X}$ denote the mean of a random sample of size n_1 from $G(x;\theta_1,\kappa)$, and independently, $\overline{Y}$ denote the mean of another random sample of size n_2 from $G(y;\theta_2,\kappa)$. If κ is known, a size α test of $H_0 : \theta_1 \leq \theta_2$ against $H_a : \theta_1 > \theta_2$ is obtained by rejecting H_0 if

$$\frac{\overline{X}}{\overline{Y}} \geq F_{2n_1\kappa,2n_2\kappa}(\alpha)$$

where $F_{\nu_1,\nu_2}(\alpha)$ denotes the upper αth percentile of the F-distribution with $\nu_1,\ \nu_2$ degrees of freedom.

When κ is unknown (Shiue and Bain, 1983), an approximate test is obtained by replacing κ by $\hat{\kappa}$ in the above test procedure, where $\hat{\kappa}$ is the MLE of κ based on the combined sample of x's and y's. Although, this makes the true significance level of the test somewhat different from the nominal level for small $n_1,\ n_2$, it depends on κ only slightly (and is free of θ). It is therefore possible to select an initial level β so that

$$Pr\left(\frac{\overline{X}}{\overline{Y}} \geq F_{2n_1\hat{\kappa},2n_2\hat{\kappa}}(\beta)\right) \approx \alpha.$$

As found by Grice and Bain (1980) in the one-sample case, in the two-sample case (Shiue and Bain, 1983) also the actual size of the test is very close to its limiting value as $\kappa \to \infty$. Conveniently, the same table of β values that is used in the one-sample case to obtain a test of actual size α can be used in the two-sample case by replacing n by $n_1 + n_2$.

Now we consider the case of $m \geq 3$ gamma distributions. Let independent random samples of sizes n_i are available from m gamma distributions $G(\cdot;\theta_i,\kappa)$ with corresponding sample means as $\overline{X}_i,\ i = 1,\cdots,m$. The problem of testing $H_0 : \theta_1 = \cdots = \theta_m$ against $H_a - H_0$ where $H_a : \theta_1 \geq \cdots \geq \theta_m$ may be considered as the conjunction of $m-1$ nested problems of testing $H_{0i} : \theta_1 = \cdots = \theta_i = \theta_{i+1}$ versus $H_{ai} : \theta_1 = \cdots = \theta_i > \theta_{i+1}$ for $i = 1,\cdots,m-1$.

For $i = 1,\cdots,m-1$, the test of H_{0i} versus H_{ai} can be based on the work of Shiue and Bain (1983) as follows. Let

$$F_i = \frac{\sum_{j=1}^{i} n_j \overline{X}_j}{N_i \overline{X}_{i+1}}$$

where $N_i = \sum_{j=1}^{i} n_j$. We reject H_{0i} if $F_i \geq F_{2N_i\hat{\kappa}_i,2n_{i+1}\hat{\kappa}_i}(\alpha)$ where $\hat{\kappa}_i$ is the MLE of κ obtained in the ith step by combining all related $i+1$ samples.

It can be shown that

$$F_i = \frac{S_1 + \cdots + S_i}{S_{i+1}} \left(\frac{n_{i+1}}{N_i}\right)$$

where $S_i \sim G(\cdot;\theta_i, n_i\kappa)$ for $i = 1,\cdots,m-1$.

Lemma 1 (Mudholkar *et al.*, 1993) Let $V_1, V_2, \cdots, V_n$ be independent gamma random variables with the same scale parameter. Then the random variables

$$W_2 = \frac{V_2}{V_1},\ W_3 = \frac{V_3}{V_1 + V_2}, \cdots, W_n = \frac{V_n}{V_1 + V_2 + \cdots + V_{n-1}}$$

are mutually independent.

Using Lemma 1 and under H_0, it follows that the statistics $F_1, \cdots, F_{m-1}$ are mutually independent.

Let $P_i(\hat{\kappa}_i) \approx Pr(F_{2N_i\hat{\kappa}_i,2n_{i+1}\hat{\kappa}_i} \geq F_i)$ denote the (approximate) p-value associated with the test statistic $F_i,\ i = 1,\cdots,m-1$. The Fisher's method of combining independent p-values is based on the test statistic

$$\psi_F(\hat{\kappa}_1, \cdots, \hat{\kappa}_{m-1}) = -2 \sum_{i=1}^{m-1} \ln P_i(\hat{\kappa}_i)$$

which has a χ^2_{2m-2} distribution (approximately) under H_0. Thus we reject H_0 if $\psi_F(\hat{\kappa}_1, \cdots, \hat{\kappa}_{m-1}) \geq \chi^2_{2m-2}(\alpha)$ for given α.

The actual size of this test is given by

$$P(\kappa, \alpha) = Pr\left[\psi_F(\hat{\kappa}_1, \cdots, \hat{\kappa}_{m-1}) \geq \chi^2_{2m-2}(\alpha)\right].$$

Under H_0, this probability does not depend on θ_i's as the joint density of F_i and $\hat{\kappa}_i$ is free of θ_i's (since the distribution of F_i is free of θ_i's and $\hat{\kappa}$ depends on the gamma variates only through A/M, where A=arithmetic mean, and G=geometric mean, and thus its distribution is free of θ_i's). In Section 3, we have obtained closed form expressions for $P(\kappa, \alpha)$. For $m = 3$, we have estimated values of $P(\kappa, \alpha)$ for various values of κ, α, n_i using Monte Carlo simulation (see Section 5) with these results and these are reported in Table 1. The case of the limiting values

$$P(0, \alpha) = \lim_{\kappa \to 0} P(\kappa, \alpha), \text{ and, } P(\infty, \alpha) = \lim_{\kappa \to \infty} P(\kappa, \alpha)$$

are derived in Section 4 and are also included in Table 1.

3 Actual Size of the Proposed Test

It is of interest to investigate the actual size of the proposed test especially for small sample sizes. To do this one can randomly generate m samples of size n_i from identical $G(x; \theta, \kappa)$ populations, compute $P_i(\hat{\kappa}_i)$, $1 \leq i \leq m-1$ and $\psi_F(\hat{\kappa}_1, \cdots, \hat{\kappa}_{m-1})$, then determine whether $\psi_F(\hat{\kappa}_1, \cdots, \hat{\kappa}_{m-1}) \geq \chi^2_{2m-2}(\alpha)$. Repeat this process a large number of times to estimate the actual size of the test. In the following, we derive a closed form expression of the actual size of the test.

We consider the case of $m = 3$ and study the actual size of the proposed test for different combinations of the (small) sample sizes and values of the shape parameter. Since the statistics $F_1,\ F_2$ are independent of $\hat{\kappa}_1,\ \hat{\kappa}_2$ (which follows from the fact that for a specified κ, F_1, F_2 are functions of the complete sufficient statistics and the distribution of $\hat{\kappa}_i$'s are free of θ's), the actual size of the test is given by

$$\begin{aligned}
P(\kappa,\alpha) &= Pr\left[\psi_F(\hat{\kappa}_1,\hat{\kappa}_2) \geq \chi^2_4(\alpha)\right] \\
&= \int_0^\infty \int_0^\infty Pr\left[\psi_F(\hat{\kappa}_1,\hat{\kappa}_2) \geq \chi^2_4(\alpha) | \hat{\kappa}_1 = a_1, \hat{\kappa}_2 = a_2\right]\ f_{\hat{\kappa}_1,\hat{\kappa}_2}(a_1,a_2)\ da_1\ da_2 \\
&= \int_0^\infty \int_0^\infty Pr\left[\psi_F(a_1,a_2) \geq \chi^2_4(\alpha) | \hat{\kappa}_1 = a_1, \hat{\kappa}_2 = a_2\right]\ f_{\hat{\kappa}_1,\hat{\kappa}_2}(a_1,a_2)\ da_1\ da_2 \\
&= \int_0^\infty \int_0^\infty Pr\left[\psi_F(a_1,a_2) \geq \chi^2_4(\alpha)\right]\ f_{\hat{\kappa}_1,\hat{\kappa}_2}(a_1,a_2)\ da_1\ da_2
\end{aligned} \tag{2}$$

where $f_{\hat{\kappa}_1,\,\hat{\kappa}_2}$ is the joint density of $(\hat{\kappa}_1,\ \hat{\kappa}_2)$.

Since

$$F_1 = \frac{\overline{X}_1}{\overline{X}_2} \sim F_{2n_1\kappa,2n_2\kappa},\ \text{and},\ F_2 = \frac{n_1\overline{X}_1 + n_2\overline{X}_2}{N_2\overline{X}_3} \sim F_{2N_2\kappa,2n_3\kappa}$$

we can express these random variables as

$$F_1 = \mathcal{F}^{-1}_{2n_1\kappa,2n_2\kappa}(U_1),\ \text{and},\ F_2 = \mathcal{F}^{-1}_{2N_2\kappa,2n_3\kappa}(U_2)$$

where $\mathcal{F}_{\nu_1,\nu_2}(\cdot)$ is the cumulative distribution function of the F distribution with ν_1,ν_2 degrees of freedom and U_1, U_2 are independent $U(0,1)$ random variables.

When $\hat{\kappa}_i = a_i$, the component p-values can also be expressed as

$$P_i(a_i) \stackrel{\mathcal{D}}{=} 1 - \mathcal{F}_{2N_ia_i,2n_{i+1}a_i}(\mathcal{F}^{-1}_{2N_i\kappa,2n_{i+1}\kappa}(U_i)) \tag{3}$$

for $i = 1, 2$. When H_0 is true, it follows that

$$\begin{aligned}
Pr\left[\psi_F(a_1,a_2) \geq \chi^2_4(\alpha)\right] &= Pr\left[-2\ \ln\ P_1(a_1) - 2\ \ln\ P_2(a_2) \geq \chi^2_4(\alpha)\right] \\
&= Pr\left[P_1(a_1)P_2(a_2) \leq e^{-\chi^2_4(\alpha)/2}\right].
\end{aligned} \tag{4}$$

In derivation of the last expression in (4), the convolution formulas may be used to find the distribution of $-2\ \ln\ P_1(a_1) - 2\ \ln\ P_2(a_2)$. However we have used the elegant conditional argument as in Mudholkar *et al.* (1993). Using (3) into (4) we obtain that $Pr\left[\psi_F(a_1,a_2) \geq \chi_4^2(\alpha)\right]$ can be expressed as

$$1-\int_0^y \mathcal{F}_{2N_2\kappa,2n_3\kappa}\left(\mathcal{F}^{-1}_{2N_2a_2,2n_3a_2}\left(1-\frac{e^{-\chi_4^2(\alpha)/2}}{1-\mathcal{F}_{2n_1a_1,2n_2a_1}(\mathcal{F}^{-1}_{2n_1\kappa,2n_2\kappa}(u))}\right)\right)du \tag{5}$$

where $y=\mathcal{F}_{2n_1\kappa,2n_2\kappa}(\mathcal{F}^{-1}_{2n_1a_1,2n_2a_1}(1-e^{-\chi_4^2(\alpha)/2}))$. Substituting (5) in (2) we obtain the actual size of our test is given by

$$E_{\hat{\kappa}_1,\hat{\kappa}_2}\left[1-\int_0^y \mathcal{F}_{2N_2\kappa,2n_3\kappa}\left(\mathcal{F}^{-1}_{2N_2\hat{\kappa}_2,2n_3\hat{\kappa}_2}\left(1-\frac{e^{-\chi_4^2(\alpha)/2}}{1-\mathcal{F}_{2n_1\hat{\kappa}_1,2n_2\hat{\kappa}_1}(\mathcal{F}^{-1}_{2n_1\kappa,2n_2\kappa}(u))}\right)\right)du\right]. \tag{6}$$

For general m, the actual size of the test is given by

$$E_{\hat{\kappa}_1,\cdots,\hat{\kappa}_{m-1}}\left[1-\int_0^\infty\cdots\int_0^\infty \mathcal{F}_{2N_{m-1}\kappa,2n_m\kappa}\left(\mathcal{F}^{-1}_{2N_{m-1}\hat{\kappa}_{m-1},2n_m\hat{\kappa}_{m-1}}\right.\right.$$
$$\left.\left.\left(1-\frac{e^{-\chi^2_{2m-2}(\alpha)/2}}{P_1(\hat{\kappa}_1)\cdots P_{m-2}(\hat{\kappa}_{m-2})}\right)\right)I\,du_1\cdots du_{m-2}\right]$$

where

$$\begin{aligned} I &= 1,\ \text{if } e^{-\chi^2_{2m-2}(\alpha)/2} \leq P_1(\hat{\kappa}_1)\cdots P_{m-2}(\hat{\kappa}_{m-2}),\\ &= 0,\ \text{otherwise}\end{aligned}$$

is the indicator function (note $P_i(\hat{\kappa}_i)$ is a function of $u_i,\ 1\leq i\leq m-2$).

4 Asymptotic Results

For the two-sample case, Shiue and Bain (1983) have shown that

$$V=2(n_1+n_2)\kappa\left[\,\ln\hat{\kappa}-\phi(\hat{\kappa})\right],$$

where $\hat{\kappa}$ is the MLE of κ obtained from the combined samples and ϕ is the digamma function, converges to a chi-square distribution with n_1+n_2-1 degrees of freedom as $\kappa\to\infty$ and to a chi-square distribution with $2(n_1+n_2-1)$ degrees of freedom as

$\kappa \to 0$. Thus for general m at the ith step we have

$$\begin{aligned} V_i &= 2N_{i+1}\kappa\,[\,\ln \hat{\kappa}_i - \phi(\hat{\kappa}_i)] \xrightarrow{\mathcal{D}} \chi^2_{N_{i+1}-1} \text{ as } \kappa \to \infty, \text{ and} \\ &\qquad \xrightarrow{\mathcal{D}} \chi^2_{2(N_{i+1}-1)} \text{ as } \kappa \to 0, \end{aligned} \tag{7}$$

for $i = 1, \cdots, m-1$ where $\hat{\kappa}_i$ is the MLE of κ obtained at the ith step.

It is also known (Ahuja and Nash, 1967) that for the two-sample case as $\kappa \to \infty$ we have

$$\left(\frac{n_1 n_2 \kappa}{n_1+n_2}\right)^{1/2} \ln\left(\frac{\overline{X}_1}{\overline{X}_2}\right) \xrightarrow{\mathcal{D}} Z$$

where Z is a standard normal random variable.

Thus for general m as $\kappa \to \infty$, at the ith step, we get

$$\left(\frac{N_i n_{i+1}\kappa}{N_{i+1}}\right)^{1/2} \ln\left(\frac{\sum_{j=1}^{i} n_j \overline{X}_j}{N_i \overline{X}_{i+1}}\right) \xrightarrow{\mathcal{D}} Z_i \tag{8}$$

for $i = 1, \cdots, m-1$ where Z_i are independent standard normal random variables.

We consider the case of $m = 3$. Using (8), when $\kappa \to \infty$, it may be shown that

$$P_i(\hat{\kappa}_i) \stackrel{\mathcal{D}}{=} 1 - \Phi\left(\left(\tfrac{N_{i+1}}{v_i}\right)^{1/2} Z_i\right)$$

for $i = 1, 2$ where Φ is the cumulative distribution function of the standard normal distribution and v_i is the observed value of V_i.

If the density function of V_i is $g_i(v_i)$, $i = 1, 2$ then using (7) we get

$$\begin{aligned} P(\infty, \alpha) &= \lim_{\kappa\to\infty} Pr\left[\psi_F(\hat{\kappa}_1, \hat{\kappa}_2) \geq \chi^2_4(\alpha)\right] \\ &= \int_0^\infty \int_0^\infty Pr\left[\left(1 - \Phi\left(\left(\frac{N_2}{v_1}\right)^{1/2} Z_1\right)\right) \cdot \right. \\ &\qquad \left. \left(1 - \Phi\left(\left(\frac{N_3}{v_2}\right)^{1/2} Z_2\right)\right) \leq e^{-\chi^2_4(\alpha)/2}\right] g_1(v_1)\, g_2(v_2)\, dv_1 dv_2 \\ &= 1 - \int_0^1 \int_0^1 \int_0^{u_0} \Phi\left(\left(\frac{\mathcal{G}^{-1}(u_2, N_3-1)}{N_3}\right)^{1/2} \cdot \right. \end{aligned}$$

$$\Phi^{-1}\left(1-\frac{e^{-\chi_4^2(\alpha)/2}}{1-\Phi\left(\left(\frac{N_2}{\mathcal{G}^{-1}(u_1,N_2-1)}\right)^{1/2}\Phi^{-1}(u_3)\right)}\right)\Bigg) \; du_3du_2du_1 \quad (9)$$

where $u_0 = \Phi\left(\left(\mathcal{G}^{-1}(u_1,N_2-1)/N_2\right)^{1/2}\Phi^{-1}\left(1-e^{-\chi_4^2(\alpha)/2}\right)\right)$, and $\mathcal{G}(\cdot,N_i)$ is the cumulative distribution function of the chi-square distribution with N_i degrees of freedom. A similar expression can be derived in the case of more general m.

For the two-sample case when $\kappa \to 0$ we have (Ahuja and Nash, 1967)

$$\frac{n_1n_2\kappa}{(n_1^2+n_2^2)^{1/2}}\ \ln\left(\frac{\overline{X}_1}{\overline{X}_2}\right) \xrightarrow{\mathcal{D}} W$$

where W has a double exponential distribution with density as

$$\begin{aligned} q_W(w) &= \tfrac{(n_1^2+n_2^2)^{1/2}}{n_1+n_2}\ \exp\left(\tfrac{w(n_1^2+n_2^2)^{1/2}}{n_2}\right), \qquad -\infty < w \le 0,\\ &= \tfrac{(n_1^2+n_2^2)^{1/2}}{n_1+n_2}\ \exp\left(-\tfrac{w(n_1^2+n_2^2)^{1/2}}{n_1}\right), \quad 0 < w < \infty. \end{aligned}$$

Thus for general m as $\kappa \to 0$, at the ith step, we get

$$\frac{N_in_{i+1}\kappa}{\left(N_i^2+n_{i+1}^2\right)^{1/2}}\ \ln\left(\frac{\sum_{j=1}^{i} n_j\overline{X}_j}{N_i\overline{X}_{i+1}}\right) \xrightarrow{\mathcal{D}} W_i \qquad (10)$$

where W_i are independent double exponential random variables with densities given by

$$\begin{aligned} q_{W_i}(w) &= \tfrac{\left(N_i^2+n_{i+1}^2\right)^{1/2}}{N_{i+1}}\ \exp\left(\tfrac{w\left(N_i^2+n_{i+1}^2\right)^{1/2}}{n_{i+1}}\right), \qquad -\infty < w \le 0,\\ &= \tfrac{\left(N_i^2+n_{i+1}^2\right)^{1/2}}{N_{i+1}}\ \exp\left(-\tfrac{w\left(N_i^2+n_{i+1}^2\right)^{1/2}}{N_i}\right), \quad 0 < w < \infty \end{aligned}$$

for $i = 1,\cdots,m-1$.

Let $Q_{W_i}(w)$ be the cumulative distribution function of W_i, $i = 1,\cdots,m-1$. Using (10), when $\kappa \to 0$, it may be shown that

$$P_i(\hat{\kappa}_i) \ \stackrel{\mathcal{D}}{=}\ 1 - Q_{W_i}\left(\tfrac{2N_{i+1}W_i}{v_i}\right)$$

for $i = 1, 2$.

It follows that

$$\begin{aligned}
P(0,\alpha) &= \lim_{\kappa\to 0} Pr\left[\psi_F(\hat\kappa_1,\hat\kappa_2) \geq \chi_4^2(\alpha)\right] \\
&= \int_0^\infty\int_0^\infty Pr\Bigg[\left(1-Q_{W_1}\left(\frac{2N_2W_1}{v_1}\right)\right)\cdot \\
&\qquad \left(1-Q_{W_2}\left(\tfrac{2N_3W_2}{v_2}\right)\right) \leq e^{-\chi_4^2(\alpha)/2}\Bigg]\ g_1(v_1)\ g_2(v_2)\ dv_1dv_2 \\
&= 1-\int_0^1\int_0^1\int_0^{v_0} Q_{W_2}\Bigg(\left(\frac{\mathcal{G}^{-1}(u_2,2(N_3-1))}{2N_3}\right)\cdot \\
&\qquad Q_{W_2}^{-1}\left(1-\frac{e^{-\chi_4^2(\alpha)/2}}{1-Q_{W_1}\left(\frac{2N_2Q_{W_1}^{-1}(u_3)}{\mathcal{G}^{-1}(u_1,2(N_2-1))}\right)}\right)\Bigg)\ du_3du_2du_1
\end{aligned}$$

where $v_0 = Q_{W_1}\left((\mathcal{G}^{-1}(u_1,2(N_2-1))/2N_2)\,Q_{W_1}^{-1}\left(1-e^{-\chi_4^2(\alpha)/2}\right)\right)$.

A similar expression can be derived in the case of more general m.

5 Monte Carlo Results

We have used the IMSL subroutines to generate gamma random variates. The rational approximations by Greenwood and Durand (1960) are used to calculate $\hat\kappa$ in ith step. Each simulated value in Table 1 is an average of the quantity inside square brackets in (6) based on 3,000 replications using numerical integration subroutine DQDAGS. Similar results were obtained at other combinations of κ, α, n_i's which are not reported here for brevity.

The limiting values in Table 1 are calculated using the formulas derived in Section 4 with subroutine DQAND. These results support the Monte Carlo results and can be helpful in interpolation purposes outside the range of the Monte Carlo study. From Table 1, it is seen that in each case considered the true level $P(\kappa,\alpha)$ is slightly above the prescribed level α. Also, these values are nearly constant for fixed values of α and

n_i's and are close to the limiting value $P(\infty, \alpha)$. Thus one may like to modify the test for small sample sizes so that the actual level is closer to the prescribed nominal level. Since $P(\kappa, \beta) \approx P(\infty, \beta)$ for small n_i's, an approximate α level test of H_0 versus $H_a - H_0$ is to reject H_0 if

$$\psi_F(\hat{\kappa}_1, \cdots, \hat{\kappa}_{m-1}) \geq \chi^2_{2m-2}(\beta)$$

where $P(\infty, \beta) = \alpha$.

We have used (9) to find values of β (where $P(\infty, \beta) = \alpha$) for various values of α and various combinations of the sample sizes. These are reported in Table 2. For smaller sample sizes, Table 2 can be used to obtain an accurate α level test. From Table 2 we observe that for higher α values, the β values are quite close to the targeted α values when sample sizes are 50 but for smaller α values even higher sample sizes are needed to avoid use of β.

As noted earlier, Shiue and Bain (1983) have shown that the limiting values $P(\infty, \beta)$ in the two-sample cases can be obtained from the one-sample case simply by replacing n by N_2. However, when using the Fisher's method of combining p-values, different expressions for $P(\infty, \beta)$ are obtained for different m as we have derived in Section 4.

6 Other Partial Orders

The tests proposed in this article can be easily extended to other partial orders. For $m = 4$ the *simple tree order* holds if

$$H_a : \theta_1 \geq \theta_i, \ i = 2, 3, 4$$

and there is no restriction among $\theta_i, \ i = 2, 3, 4$. To test $H_0 : \theta_1 = \cdots = \theta_4$ versus $H_a - H_0$, one first tests equality between θ_2 and θ_3 against they are not equal based on the second and third samples (that is, decide 'do not reject $\theta_2 = \theta_3$' if $F_{2n_2\hat{\kappa},2n_3\hat{\kappa}}(1-\alpha/2) \leq \overline{X}_2/\overline{X}_3 \leq F_{2n_2\hat{\kappa},2n_3\hat{\kappa}}(\alpha/2))$. Then the second and third samples are combined, and a test of hypothesis with unrestricted alternative is conducted comparing with the fourth sample. Finally a test of hypothesis is conducted between the first sample and the combined second, third and fourth samples with the restricted alternative as specified in H_a. Thus the test statistics are

$$F_1 = \frac{\overline{X}_2}{\overline{X}_3},\ F_2 = \frac{n_2\overline{X}_2 + n_3\overline{X}_3}{(n_2+n_3)\overline{X}_4},\ F_3 = \frac{(n_2+n_3+n_4)\overline{X}_1}{n_2\overline{X}_2 + n_3\overline{X}_3 + n_4\overline{X}_4}.$$

Lemma 2 (Johnson and Kotz, 1970) If $S_1, S_2, \cdots, S_k$ are independent gamma random variables having the same scale parameter, then the two random variables $\sum S_i$ and $S_j/\sum S_i$ are independent for each $j,\ j = 1, \cdots, k$.

Using Lemma 1 and Lemma 2 above, it can be shown that F_1, F_2, F_3 are independent under H_0. The corresponding p-values can therefore be combined to obtain the overall test.

The test can also be extended to more *general loop order*. For $m = 6$ consider the alternative hypothesis

$$H_a : [\theta_1, \theta_2] \geq [\theta_3, \theta_4]; \theta_5 \geq \theta_6.$$

Here the semi-colon separates the scale parameters into two groups, and there is no relation between these two groups. Also when the scale parameters are inside square brackets there are no relation among themselves . The test procedure would result in five component p-values arising from the following tests: (1) Test $H_0 : \theta_1 = \theta_2$ versus $H_a : \theta_1 \neq \theta_2$; (2) Test $H_0 : \theta_3 = \theta_4$ versus $H_a : \theta_3 \neq \theta_4$; (3) Test $H_0 : (\theta_1, \theta_2) = (\theta_3, \theta_4)$ versus $H_a : (\theta_1, \theta_2) > (\theta_3, \theta_4)$, where scale parameters within parenthesis means that the corresponding samples are combined; (4) Test $H_0 : \theta_5 = \theta_6$ versus $H_a : \theta_5 > \theta_6$; (5) Test $H_0 : (\theta_1, \cdots, \theta_4) = (\theta_5, \theta_6)$ versus $H_a : (\theta_1, \cdots, \theta_4) \neq (\theta_5, \theta_6)$. These related p-values can be shown to be independent under H_0, and hence can be combined to construct the test statistic.

However note that the procedure applies when one scale parameter appears in one and only one group. Thus an example of more general partial orders when the procedure does not apply for $m = 4$ is, e.g. $\theta_1 \geq [\theta_2, \theta_3]; \theta_3 \geq \theta_4$. Also the procedures described are not order invariant, that is the power of the overall test depends on the order in which the component tests are carried out.

7 Concluding Remarks

Tests for gamma distribution for $m \geq 3$ have been treated in literature very rarely. We have proposed a test of equality against partial orders of scale parameters of $m \geq 3$ gamma distributions and studied its small sample properties under the null hypothesis. The test can be implemented for any $m \geq 3$ using formulas derived in this paper and is applicable in a variety of order restrictions with any sample sizes. The procedure is simple and easy to implement for moderate to large sample sizes. For small sample sizes, in general, one requires numerical integration to obtain a size α test. The approximation error is negligible for small to moderate sample sizes. The asymptotic distribution of the proposed test is not of the chi-bar squared type which are difficult as they depend on (usually intractable) level probabilities.

Acknowledgment

The author thanks the referee for careful reading and many comments which helped to improve the presentation in this paper.

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Table 1: Actual size comparison for $m = 3$ case

				α						α	
n_1	n_2	n_3	κ	.05	.10	n_1	n_2	n_3	κ	.05	.10
5	5	5	0	.086	.143	10	10	5	0	.069	.124
			.1	.076	.130				.1	.063	.114
			1.0	.081	.134				1.0	.066	.118
			4.0	.081	.134				4.0	.066	.118
			∞	.083	.137				∞	.067	.119
5	5	10	0	.080	.135	10	5	10	0	.074	.130
			.1	.071	.123				.1	.066	.118
			1.0	.076	.129				1.0	.069	.122
			4.0	.078	.131				4.0	.070	.122
			∞	.080	.133				∞	.070	.123
5	10	5	0	.072	.126	5	10	10	0	.068	.121
			.1	.065	.117				.1	.062	.113
			1.0	.069	.122				1.0	.067	.119
			4.0	.071	.123				4.0	.069	.121
			∞	.072	.125				∞	.070	.123
10	5	5	0	.078	.135	10	10	10	0	.067	.121
			.1	.070	.123				.1	.060	.112
			1.0	.072	.125				1.0	.065	.117
			4.0	.072	.125				4.0	.064	.115
			∞	.072	.125				∞	.065	.118

Table 2: Starting choice of β to obtain actual size α for $m = 3$

			α						
n_1	n_2	n_3	.0075	.01	.025	.05	.075	.10	.25
5	5	5	.0008	.0014	.0077	.0237	.0432	.0650	.2153
5	5	10	.0010	.0018	.0088	.0258	.0461	.0683	.2191
5	10	10	.0022	.0034	.0128	.0324	.0544	.0775	.2286
10	10	10	.0031	.0046	.0152	.0361	.0589	.0827	.2336
20	20	20	.0050	.0070	.0198	.0428	.0669	.0913	.2419
30	30	30	.0058	.0079	.0215	.0452	.0695	.0943	.2447
50	50	50	.0064	.0086	.0229	.0471	.0717	.0965	.2469
∞	∞	∞	.0075	.0100	.0250	.0500	.0750	.1000	.2500