

The Road Coloring Problem, the Algebra Project, and the Writing of K-12 Curricula Based on “Mathematically Rich Experiences”

by

Gregory Budzban
Department of Mathematics, Southern Illinois University
Carbondale, IL 62901

“What are you working on?”

A rather innocent question. I’m never quite sure how to answer when it comes from a non-mathematician. Should I answer in some detail, and wait for the inevitable glazed-eyes and uneasy expression that signals it is time to stop. Or should I answer in vague and imprecise terms, and risk having the listener feel that I am patronizing them.

But this time the question came from Robert Moses, renowned civil-rights activist, key organizer of the Mississippi voting rights project, and MacArthur Fellow. More recently his work has been as founder and head of the Algebra Project, an organization dedicated to a national mathematics literacy effort so that low income and, in particular, African American and Latino/a students can gain the mathematical skills necessary to succeed in today’s increasingly technological society.

As the Math Department Colloquium Coordinator for Southern Illinois University Carbondale, I had invited Robert Moses to campus to give a talk on his work and was acting as his host while he was at the university. After an extremely busy visit, we had stopped for lunch on the way to the airport. He posed the question as we talked over coffee after the meal.

I grabbed a pen and a napkin and began to describe the Road Coloring Problem.

The Road Coloring Problem (RCP) first appeared in published form in the 1977 paper of Adler, Goodwyn, and Weiss [AGW], entitled “*Equivalence of topological Markov shifts*”, though the problem was implicit in earlier work [AW]. Within the framework of symbolic dynamics, investigations led to the following question.

Suppose one is given a strongly connected, directed graph. For simplicity, assume that the out-degree of each vertex is two. A “road-coloring” of this graph is an assignment of two labels, say r and b , to the edges in such a way that each vertex has exactly one edge labeled r and one labeled b leading away from it. A given graph will have many such road-colorings.

Once a graph is labeled in this way, any finite sequence of r ’s and b ’s can be thought of as a transformation on the set of vertices of the graph. A *synchronizing instruction* is

such a sequence that acts as a constant transformation. Colloquial descriptions of the problem often talk about thinking of each vertex as a building in a city, and a synchronizing instruction as a set of directions that leads everyone in the city, regardless of which building they are in initially, to the same building at the same time. (See Figure 1.)

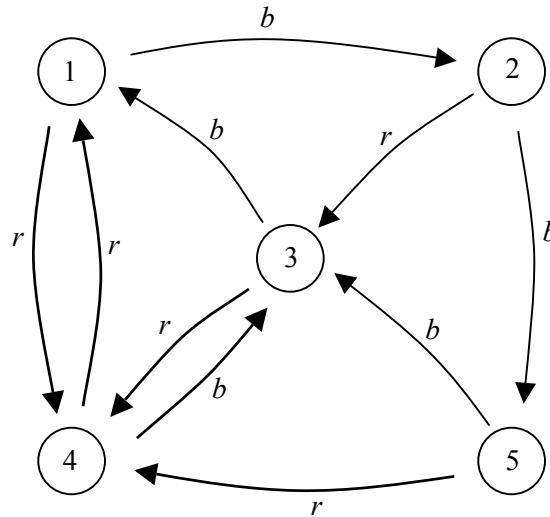


Figure 1. A road coloring of a strongly connected, aperiodic digraph. The instruction $brbr$ maps all vertices to vertex 4.

The RCP arose from trying to understand the connection between having a synchronizing instruction and the periodicities within the graph. Here the *period* of a vertex is the greatest common divisor of the cycle lengths for that vertex, where a *cycle* is a sequence of edges that start and end at the given vertex. It is easy to show that if the graph is strongly connected all of the vertices have the same period, so one can speak of the period of the graph itself. In the original paper, Adler and his colleagues showed that for a given graph, the existence of a road-coloring with an inherent synchronizing instruction proves that the graph had to have period one, that is, the graph is *aperiodic*. Thus aperiodicity is *necessary* for a strongly connected graph to have a road-coloring with an inherent synchronizing instruction. The RCP asks if aperiodicity is *sufficient*, as well. The references below point to special cases of the problem, and its generalization for periodic graphs, that have been proven over the years.

Moses listened intently, asked some pointed questions to clarify an idea or two, and then sat back. He asked if he could take the napkins with him, and I was only too happy to oblige. I dropped him off at the airport, shook his hand, and told him what an honor and privilege it was to have met him. In a completely unassuming way, he thanked me for inviting him and, carrying his luggage, walked into the airport.

Two weeks later, I found a cryptic message in my departmental mailbox from Ben Moynihan, who coordinates Algebra Project initiatives from their national office in Cambridge, MA, telling me I should call Jackson, Mississippi because Bob Moses

wanted to talk with me about the Road Coloring Problem and a possible NSF proposal. I had no idea what this message was talking about. While I had great admiration for his work, I knew he was not doing any active mathematical research, and his daily classroom teaching activities at Lanier High School in Jackson would seem to preclude any time to begin a research program.

When I called, Moses explained to me that when he had returned to the classroom, he had showed his students the RCP and challenged them to find synchronizing instructions. His students rose to the challenge and began to construct “cities” and search out the different ways to color the roads. They loved “solving the puzzle” and were fascinated by the unsolved nature of the problem. They had never heard of a math problem for which no one knew the answer. He said he wanted to submit an NSF proposal to write some high school curriculum around the problem, and wanted to know if I had ever considered what high school level mathematics could be extracted from what he called the “experience”.

By now I was intrigued and drove to Jackson to see for myself. The students at Lanier High School, near downtown Jackson, are almost exclusively African-American. Each morning they walk through metal detectors in front of numerous uniformed police officers on their way to classes. It was the middle of May when I first visited and the summer heat was also oppressive. Old air-conditioning vents leaked water from the ceilings into rusted trash cans strategically placed in the hallways. Bob met me and said that he was going to bring all of his 9th grade students to the library where I could meet them and watch them working on the problem. He hoped I could talk to them a little about my research and describe some the mathematics involved. As I waited in the library for the students to arrive, I wondered what in the world I could say.

The students, more than a hundred, filed into the library and sat in groups of six to eight at large tables. They brought with them large sheets of paper on which “cities” had been drawn, and boxes of poker chips. I wondered what the poker chips were for. Bob introduced me, and told them that I was the mathematician who had described the problem to him, and then stepped back.

I told them that I was glad to be at Lanier and happy to hear that they were interested in working on the problem. I said that I would talk a little about my work on the problem and some of the math involved, but first I wanted to see what they had already learned about it. I was really stalling for time.

Bob stepped up, and told the students that each table should take the cities they had designed and try to find the solution for them. I watched as the students laid the large sheets of chart paper on the tables in front of them, and placed poker chips in each of the circles. Seemingly spontaneously, one of the students at a table would say something like, “Try this..” and then begin to say a list of instructions, “Red-Red-Blue...”. The other students at the table would help to move the poker chips around the digraph. It was wonderful to watch.

I walked between the tables and tried to observe unobtrusively. They were aware of my presence, but it didn't seem to make them any less attentive to the task. Seeing over a hundred 9th grade students working on the RCP was an extraordinary experience. Each table worked hard at the problem. Almost invariably they found a solution, and then would carefully check it to make sure it worked. I saw students working with graphs having ten or twelve vertices--much bigger than the examples I would usually work with—their fingers flying around the chart paper moving the poker chips.

As I watched and listened, I heard conversations emerge that provided some insight into their thought processes. They talked about the chips, “moving together” and “going to the same place”. They spoke of the buildings they were “leaving from” and the buildings they were “going to”. They almost always moved the pieces at the same time, coordinating their actions with their fellow tablemates, so that when, say, “Red” was called, all the chips “jumped” at the same time. I realized what Bob meant when he referred to an “experience”. The students were having a first hand, physical experience of *functions and their compositions*.

The set of all functions on a finite set, of course, forms a semigroup under composition. These finite transformation semigroups are wonderful structures of seemingly infinite variety. All finite semigroups can be realized as subsemigroups of finite transformation semigroups, using exactly the same construction as in Cayley's theorem in finite group theory, though occasionally an identity element must be added to the finite semigroup to ensure the representation is faithful. (An example for when this would be necessary would be the two element semigroup with $00=10=01=11=0$. Adjoining an identity element, say e , will produce distinct transformations to represent 0 and 1.) The diversity of finite semigroups, however, refuses to yield to any classification program as was done for finite groups.

One class of finite semigroups for which a structure theory exists are those which are *simple*, having no two-sided ideals other than the semigroup itself. In this case, a full understanding of the structure comes from the following construction. Let S be a finite simple semigroup and let $E(S)$ be the set of idempotents of S . Impose on $E(S)$ the order $e \leq f$ iff $ef = fe = e$. Choose an idempotent $e_0 \in S$ minimal with respect to this order (finiteness assures its existence) and let

$$X = E(Se_0) \quad G = e_0Se_0 \quad Y = E(e_0S).$$

It can be shown that G becomes group with identity e_0 , X becomes a left-zero semigroup, (that is, each element in X acts like a zero when placed on the left) and Y becomes a right-zero semigroup. The set $X \times G \times Y$ becomes a semigroup when given the product,

$$(x_1, g_1, y_1)(x_2, g_2, y_2) := (x_1, g_1(y_1x_2)g_2, y_2).$$

Notice that $y_1x_2 \in G$. The mapping

$$\phi : Y \times X \rightarrow G \quad \phi(y, x) = yx$$

contains important information about the structure of S and is called the *sandwich function*. In particular, if $\phi(y, x) = e_0$ for all $(y, x) \in Y \times X$, then it is easy to show that $X \times G \times Y$ is a direct product.

This product structure is called the Rees product for S , and it is an important result that the mapping

$$\varphi : X \times G \times Y \rightarrow S \quad \varphi(x, g, y) = xgy$$

is an isomorphism. In general, a simple semigroup with a minimal idempotent is called completely simple and the construction extends to such cases even when the semigroup is infinite. The above theory can be found in any standard reference on semigroups and a list of several is provided. [CP] [L] [P]

When S is a finite simple transformation semigroup, the sets X , G , and Y can be further characterized. Suppose S is some subsemigroup of functions on the set $\{1, \dots, n\}$. Then each idempotent $e \in X$ can be associated with a partition π_e of the set $\{1, \dots, n\}$ in the standard way: $\pi_e = \{e^{-1}(j) : 1 \leq j \leq n\}$. Thus the set X can also be thought of as a collection of partitions of the set $\{1, \dots, n\}$. Similarly each idempotent $f \in Y$ can be associated with its range $R_f = \{jf : 1 \leq j \leq n\}$, where the convention of writing inputs to the transformation on the left is employed. The group $G = e_0 S e_0$ can then be seen as the subgroup of transformations within S having the same partition and range as e_0 .

When S is such that $X = E(S e_0)$ is a single idempotent, it is easy to show that S is right simple, having no nontrivial right ideals, and left cancellative. In this case, S is referred to as a right group and its Rees product is of the form $\{e_0\} \times G \times Y$. Equivalently, a right group of transformations is a semigroup of transformations all of which have the same partition.

Now in the case of the RCP, once you have a road-coloring of the graph, the two transformations, perhaps r and b again, generate a transformation semigroup. My colleagues and I have taken to calling these coloring semigroups and studying the entire collection of coloring semigroup associated with a given digraph. These coloring semigroups will rarely be simple, but since each finite semigroup has a minimal ideal, and this minimal ideal is necessarily simple, the above construction will hold. When a coloring semigroup has a synchronizing instruction the minimal ideal can be shown to be a right group. Several recent papers [B], [CKK], [BF] have emphasized the pivotal nature of right groups for the RCP. In fact, the results of [CKK] can roughly be stated that if some road-coloring produces a semigroup whose minimal ideal is a right group,

then either that semigroup contains a synchronizing instruction, or a “re-coloring” of the graph will.

This theory is quite beautiful, and the use of semigroups in settings as far ranging as probability, functional analysis, and automata theory is well known. What portion of this theory is accessible to 9th graders and how can it help build the mathematical background they need to succeed in their educational careers? These were the questions I asked myself, walking around the library observing the enthusiastic investigations.

I noticed that some of the digraphs that the students were working with had the property that each vertex also had an in-degree of two. This structure permitted a road-coloring in which both r and b were permutations which would generate a group. Such a coloring would never produce the constant transformation necessary for a synchronizing instruction. I listened as they struggled with this problem, and had an idea of what to discuss.

I went to the front of the library, asked them to stop what they were doing, and on the left side of a sheet of chart paper drew a digraph having a road-coloring of two permutations. Next to this I took the same digraph and road-colored it with two transformations that would produce a synchronizing instruction. I asked the students if they could see a difference in the two road-colorings and if they could explain what the difference was. Almost immediately, a young man near the front raised his hand. “You could never solve the puzzle with the one on the left”, he said confidently. “Why”, I asked, “Could you explain your reasoning to me?” “Each person goes to a separate building. You never have two people going to the same building, so you’ll never be able to get everyone together. In the other one, you can get two people to go to the same building.”

Bob immediately stepped forward, and asked the student, to turn around to the entire group and repeat what he had said, so that everyone could hear. (I later learned how important it was to Bob to nurture in the students the ability to articulate their ideas to a group.) The student repeated what he had said, and Bob asked the class if other people agreed. I saw many students nodding their heads. This student, it seemed, had clearly distinguished the difference between one-to-one and many-to-one transformations, and could express the basic differences clearly in his own words.

I looked over at Bob, who was standing to the side with his arms folded, smiling. I nodded to Bob, and said to the student, “That is exactly right. You could never solve the puzzle with the coloring on the left.” I knew at that moment that I would soon begin to write a 9th grade curriculum based on the Road Coloring Problem.

Thus began my collaboration with Robert Moses and the Algebra Project. Over the next several months, together with other members of the Algebra Project, we put together a proposal to write a new 9th grade curriculum based in part on the RCP and also drawing from Moses’ and others work on curriculum based on “mathematically rich” experiences. There was also a strong technology component to the proposal, which drew upon the work of Alan Shaw, whose PhD from MIT while working under Papert, involved the construction of internet tools that could be used by schools for the

community building process that is essential to an Algebra Project site. The importance of community organization was a direct result of Bob's civil rights work and is explained in detail in his book with Charles Cobb, *Radical Equations, Civil Rights from Mississippi to the Algebra Project* [MC]. The proposal we wrote, entitled "Raising the Floor" was eventually funded, and the curriculum that resulted is now being piloted in high schools in Chicago, Rochester, NY, Irvington, NJ, and at Lanier.

As the curriculum project develops, other mathematicians have gotten involved. Ed Dubinsky, a member of the NSF advisory panel for "Raising the Floor", saw the RCP material and had a chance to talk with Moses at length about the philosophical basis for the Algebra Project pedagogy and became involved in a major way. His involvement with Bob Moses' work began forty years ago in Mississippi during the civil rights struggles. He has co-authored with Bob a curriculum module based on the "trip line" material that is described in *Radical Equations*. Dubinsky's long career in both mathematics and math education is well known.

David Henderson of Cornell became involved after reading *Radical Equations* and meeting Moses, and Frank Davis and Mary West of Lesley University, at the NSF conference "Culturally Responsive Mathematics Curricula" in November 2004. In typical fashion, Bob approached Henderson with a mathematical question. A recent joint grant, secured by Guy Johnson of Rochester Institute of Technology from the GE Foundation's Math Excellence Fund to begin development of a 10th grade Geometry curriculum, provided the perfect opportunity for Henderson to get involved with a project that can make use of his considerable background in the area. Staffas Broussard from the University of New Orleans and William Crombie, who have been involved in Algebra Project curriculum design for many years, continue to make invaluable contributions and are also involved in the design of the Geometry curriculum, along with Dubinsky and myself.

What has attracted these people to the work of the Algebra Project? I would argue that a necessary condition is an awareness of the extent of the problem in math education at the K-12 level and its particular impact on minority groups in this country. There is also a need for a belief that, whatever your side in the "math wars" (more on these below), the current K-12 curriculum needs revision. I firmly believe that no one who has carefully looked at the majority of the current high school textbooks could come away with anything but this belief. Perhaps most importantly, one should believe that the ability to do mathematics, like the ability to read and write, is not a property of some elite component of human society but a capability that is within reach of the majority of human beings.

Let me clarify. Only a small segment of society makes their living by being professional writers. This does not stop the vast majority of people from being able to write and read. We can lament that they do not read more, or write better, et cetera, but the percentage of the population that has a basic literacy is large.

Contrast this with the corresponding level of “mathematical literacy” one would want in today’s society. Now, it is clear that only a small segment of society will be drawn to, and engage in, the process of proving new theorems. But the almost complete ignorance of mathematics that is apparent in a large percentage of the population is a national tragedy.

In addition to believing that some type of reform should occur, mathematicians who want to get involved in this work should have a research problem that can be “experienced”. In other words, the essential nature of the problem is manifested in some physical experience that the students can reconstruct. Of course, the mathematics that emerges from the experience should be accessible to students at the K-12 level and should be sufficiently mainstream that it leads to the background one needs to be successful at the college level.

The physical experience is central to Algebra Project curricula, but the process only begins there. A detailed explanation of the Algebra Project “Five Step Process” is contained in *Radical Equations*, and any serious discussion of it would require an article in itself. In brief, it is a process that begins with the physical experience, flows through the use of natural language descriptions of the event by the students (“people talk”), and finishes with abstract and symbolic representations of the mathematical features of the event.

Any curriculum process of this sort must be aware of the “math wars”. Again, any detailed, that is to say nuanced, explanation of the position of the Algebra Project would be too long to include here. Personally, I am a centrist in this oftentimes uncivil “conversation”. Students need both conceptual understanding and the ability to perform (without the aid of technology) the algorithms and procedures that we refer to as symbolic manipulation. Students must be prepared for the various state and national tests that have become increasingly important for their educational careers. There is no other short term option. The curriculum I have contributed to the Algebra Project tries, to the best of my ability, to maintain a balance between these sometimes competing interests.

For me, another part of the motivation for the work is to give students a sense of the creative and aesthetic nature of mathematical research. Much of what passes for math curriculum at the K-12 level is the equivalent of grammar and vocabulary in a natural language. Imagine if, for twelve years, all you were taught was grammar and vocabulary in your native language. Imagine that you never had the opportunity to read anything other than the “grammar book” and all you did was diagram sentences and write definitions. This is the equivalent of what we do in mathematics education at the K-12 level, in my opinion.

For a true national mathematical literacy effort an important question is, “How does one keep students involved and motivated?” For this, it is imperative that we display to students the creative and aesthetic nature of our beautiful subject. Jerry P. King wrote eloquently concerning this in his book, *The Art of Mathematics* [Ki]. Yes, I understand students need the basics and a good foundation. My response to those who would

emphasize only the basics, only the three “R’s” above all else is, “Two of the three R’s are misspelled.”

We can do better. We must do better.

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