CONVOLUTION PRODUCTS OF PROBABILITY MEASURES ON FINITE COMPLETELY SIMPLE SEMIGROUPS

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Theorem 1. *Let $S = X \times G \times Y$ be a discrete completely simple semigroup. Let $(\mu_n) \in P(S)$. Then $\mu_n(Sx)$ converges for all x as $n \to \infty$ iff $\mu_{k,n}$ converges as $n \to \infty$ on the set of all H-classes of S.*

Proof. Let $\mu_{k,n_i} \to v_k$, $\mu_{k,p_i} \to v'_k$ weakly. Now

$$\mu_{k,n_i}(Hx) = \mu_k(xS)\mu_{n_i}(Sx)$$
$$\mu_{k,p_i}(Hx) = \mu_k(xS)\mu_{p_i}(Sx).$$

Since $\mu_n(Sx)$ converges for all x, as $i \to \infty$, $v_k(Hx) = v'_k(Hx)$. Hence, for each k, all the limit points of μ are identical on the H-classes of S, therefore on S'.

Now consider

$$v_k * v_k(Hx) = \int v_k[(Hx)z^{-1}]v_k(dz).$$

Notice that

$$\begin{aligned}(Hx)z^{-1} &= (xS \quad \text{if} \quad z \in Sx \\ &= (\ \emptyset \qquad\quad z \notin Sx.\end{aligned}$$

Thus

$$\begin{aligned}v_k * v_k(Hx) &= v_k(xS)v_k(Sx) \\ &= \mu_k(xS)\lim_{n\to\infty}\mu_n(Sx) \\ &= v_k(Hx)\end{aligned}$$

since

$$v_k(xS) = \lim_{n\to\infty}\mu_{k,n}(xS) = \mu_k(xS)$$

and

$$v_k(Sx) = \lim_{n\to\infty}\mu_{k,n}(sx) = \lim_{n\to\infty}\mu_n(Sx).$$

Now suppose $\mu_{k,n}(Hx)$ converges for all k

$$\begin{aligned}\mu_{k,n}(Hx) &= \int \mu_{k,n-1}(Hx)z^{-1}\mu_n(z) \\ &= \mu_{k,n-1}(xS)\mu_n(Sx) \\ &= \mu_k(xS)\mu_n(Sx).\end{aligned}$$

□

Theorem 2. *Let $(\mu_n) \subset P(S)$, where $S = X \times G \times Y$ is a finite completely simple semigroup. Suppose $\mu_{k,n} \to v_k$ weakly for some k. Then there does not exist a normal subgroup G' of G such that $YX \subset G'$ with $S\mu_i \subset X \times g_i G \times Y$ where $g_i \in G - G'$ infinitely often.*

Proof. Suppose there did exist a normal subgroup $G' \subset G$ that satisfied the above property. Then

$$\begin{aligned}\operatorname{supp}(\mu_i * \mu_{i+1}) &= \operatorname{supp}(\mu_i)\operatorname{supp}(\mu_{i+1}) \\ &\subset (X \times g_i G' \times Y)(X \times g_{i+1} G' \times Y) \\ &= X \times g_i g_{i+1} G' \times Y\end{aligned}$$

and in general,

$$\operatorname{supp}(\mu_{i,n}) \subset X \times \prod_{k=i+1}^{n} g_k G' \times Y.$$

Now assume that for some k, $\mu_{k,n} \to v_k$ as $n \to \infty$. Consider $\prod_{j=k}^{\infty} g_j$. This product can not converge since for all j, $e \neq g_j \in G - G'$. (In a discrete group $\prod_{j=1}^{\infty} g_j$ converges iff there exists an N such that for all $n > N$, $g_j' = e$). Hence, there exists limit points h, h', $h \neq h'$. Thus for infinitely many n, $\operatorname{supp}(\mu_{k,n}) \subset X \times hG' \times Y$, and for infinitely many n, $\operatorname{supp}(\mu_{k,n}) \subset X \times h'G' \times Y$. But if $\mu_{k,n}$ converges, this is impossible. □

We will now construct a quotient structure on S and define a type of convergence relative to this structure. Let $S = X \times G \times Y$ be a finite completely simple semigroup and suppose G' is a normal subgroup of G. Assume also that $YX \subset G'$. Consider the set $S/G' = (X \times gG' \times Y : g \in G)$. Let g', g'' be elements of G. Then

$$\begin{aligned}(X \times g'G' \times Y)(X \times g''G' \times Y) &= (X \times g'G'(YX)g''G' \times Y) \\ &= (X \times g'G'g''G' \times Y) && \text{since } G'(YX) = G' \\ &= (X \times g'g''G' \times Y) && \text{since } G' \text{ is normal}\end{aligned}$$

but this is an element of S/G'. In a similar manner one can show that the operation is associative. In fact, it is clear that $X \times G' \times Y$ serves as an identity for the set and given an element $X \times gG' \times Y$, $X \times g^{-1}G' \times Y$ is its inverse. Thus the set of elements forms a group. We will say that $X_k X_{k+1} \ldots X_n$ converges mod G' with probability one iff the sequence converges in S/G' pointwise with probability one.

Theorem 3. *Let $S = X \times G \times Y$ be a finite completely simple semigroup. Let (X_n) be a sequence of independent random variables with values in S and with distributions $(\mu_n) \subset P(S)$. Let G' be a normal subgroup of G such that $YX \subset G'$. Then $X_1 X_2 \ldots X_n$ converges mod G' iff $\sum_{n=1}^{\infty} \mu_n\big[S - (X \times G' \times Y)\big] < \infty$.*

Proof. Let X, Y, G' be as above and suppose $\sum_{n=1}^{\infty} \mu_n\big[S-(X\times G'\times Y)\big] < \infty$. Consider $A = \{w : X_n(w) \in S-(X\times G'\times Y) \text{ i.o.}\}$. Then by the Borel-Cantelli lemma $P(A) = \emptyset$. Thus for $w \in A$ there exists $N(w)$ such that $X_n(w) \in X \times G' \times Y$ for $n > N(w)$. Now $\prod_{i=1}^{\infty} X_i(w) = \prod_{i=1}^{N-1} X_i(w) \prod_{i=N}^{\infty} X_i(w)$. Since $YX \subset G'$, $\prod_{i=N}^{\infty} X_i(w) \subset X \times G' \times Y$. Also $\prod_{i=1}^{N-1} X_i(w) = (x, g, y) \subset X \times g'G' \times Y$ for some g' belonging to G. Hence, $\prod_{i=N}^{\infty} X_i(w) \in (X \times g'G' \times Y)(X \times G' \times Y) = X \times g'G' \times Y \in S/G'$. Thus $X_1X_2 \dots X_n$ converges mod G' with probability one. Now suppose $\sum_{n=1}^{\infty} \mu_n\big[S-(x\times G'\times Y)\big] = \infty$. This implies that $P(A) = 1$ where A is defined as above. Then if $w \in A$, there exists (n_i) such that $X_{n_i}(w) \in S-(X\times G'\times Y)$ for all i. Thus for each i, $x_i(w) \in X\times g_iG'\times Y$ for some g_i in $S - G'$. Now $\prod_{i=1}^{\infty} X_i(w) \in X \times \left(\prod_{i=1}^{\infty} g_iG'\right) \times Y$ and since $g_i \notin G'$ and S is discrete, $\prod_{i=1}^{\infty} g_i$ cannot converge. Thus $X_1X_2 \dots X_n$ does not converge mod G'. □

Theorem 4. *Let $S = X \times G \times Y$ be a completely simple semigroup. Suppose H is a subgroup of G such that $Y'X' \subset H$. Then the following results are true.*

a) *If $z \in X \times H \times Y$, then $(X \times H \times Y)z^{-1} = X \times H \times Y'$*
b) *If $z \notin X \times H \times Y$, then $(X \times H \times Y)z^{-1} \subset X \times H^c \times Y$.*

Proof. Suppose $z \in X \times H \times Y$; $z = (x, h, w)$, $h \in H$. Let $y \in (X \times H \times Y)z^{-1}$, $y = (y_1, y_2, y_3)$. Then $yz \in X\times H\times Y$. For $z \in S' = X'\times H'\times Y'$, $(X'\times H'\times Y')z^{-1} = X'\times H'\times Y$ if $YX \subset H'$. $X_1\times H\times Y$, $yz \in X_1\times H\times Y$, $(y_1y_2y_3)(z_1z_2z_3)y_2y_3z_1 \in H$, $y_2(y, z_1)z_2 \in H$.)

$$\begin{aligned}
&\Longrightarrow (y_1, y_2, y_3)(x, h, w) \in X \times H \times Y \\
&\Longrightarrow y_2(y_3x)h \in H \\
&\Longrightarrow y_2 \in H \\
&\Longrightarrow y \in X \times H \times Y.
\end{aligned}$$

Thus $(X \times H \times Y)z^{-1} \subset X \times H \times Y$ for z belonging to $X \times H \times Y$. The converse is true since $X \times H \times Y$ is a subsemigroup. Therefore (a) above has been proved.

Now suppose $z \notin X \times H \times Y$. Then $z = (x', h', w)$ where $h' \notin H$. Let $y \in (X \times H \times Y)z^{-1}$.

$$\begin{aligned}
&\Longrightarrow y_2(y_3x)h' \in H \quad \text{but} \quad (y_3x) \in H \\
&\Longrightarrow (y_3x)h' \notin H \\
&\Longrightarrow y_2 \notin H \\
&\Longrightarrow (X \times H \times Y)z^{-1} \subset X \times H^c \times Y
\end{aligned}$$

and (b) is proven. □

Theorem 5. *Let $S = X \times G \times Y$ be a completely simple semigroup. Let (X_i) be independent random variables with values in S and with distributions $(\mu_i) \subset P(S)$. Let $\mu_{k,n} \to \upsilon_k$; $\upsilon_k \to \pi_\infty$. Then if $X_1 X_2 \ldots X_n$ converges mod G', $S_{\pi_\infty} \subset X \times G' \times Y$. This is true for any subsemigroup also. Therefore $\sum \mu_n(S - S') < \infty \implies S_{\pi_\infty} \subset S'$.*

Proof. Suppose $X_1 X_2 \ldots X_n$ converges mod G' with probability one. Then $\sum\limits_{n=1}^{\infty} \mu_n\big[S - (X \times G' \times Y)\big] < \infty$. Now by the Borel-Cantelli lemma $P(A) = \emptyset$, where $A = \{w : X_n(w) \in S - (X \times G \times Y) \text{ i.o.}\}$. Thus $P(B) = 1$ where $B = A^v = \{w : X_n(w) \in X \times G' \times Y \text{ eventually}\}$. Now let $B_i = \{w : X_i(w),\ X_{i+1}(w),\ \ldots \in X \times G \times Y)\}$. Then $B_i \subset B_{i+1}$ and $B = \bigcup\limits_{i=1}^{\infty} B_i$. Notice that if

$$\begin{aligned} w \in B_k &\implies X_k(w),\ x_{k+1}(w)\ \ldots \in X \times G' \times Y && \mu_{kn_i} \to \nu_k \\ &\implies X_k X_{k+1} X_{k+2} \ldots \in X \times G' \times Y && \nu_{n_i} \to \pi_\infty \\ &\implies P(B_k) \le \upsilon_k(X \times G' \times Y) && \text{for all } k \\ &\text{but } P(B_k) \uparrow 1 \text{ as } k \to \infty && \text{thus } \pi_\infty(X \times G' \times Y) = 1 \\ &\implies S_{\pi_\infty} \subset X \times G' \times Y. \end{aligned}$$

□

Theorem 6. *Let $S = X \times G \times Y$ be a finite completely simple semigroup. Let $(\mu_n) \subset P(S)$. Suppose $\mu_{k,n} \to \upsilon_k$ for all k and let $\upsilon_{n_i} \to \pi_\infty = \pi_\infty^2 = (\lambda_1, w_H, \lambda_2)$ where $H \subset G$. Suppose $YX \subset H$, then*

$$\sum_{n=1}^{\infty} \mu_n\big[S - (X \times H \times Y)\big] < \infty$$

and for all proper subgroups $H' \subset H$

$$\sum_{n=1}^{\infty} \mu_n\big[S - (X \times H' \times Y)\big] = \infty.$$

Proof. Suppose $\mu_{k,n} \to \upsilon_k$ for all k. Then there exists a subsequence (n_i) such that $\upsilon_{n_i} \to \pi_\infty = (u_1, w_H, u_2)$. Suppose $YX \subset H$. Now $S_{\pi_\infty} \subset X \times H \times Y$, thus given $e/4 > 0$ there exists I such that for $i > I$,

$$\upsilon_{n_i}(X \times H \times Y) > 1 - (e/4).$$

Now $\mu_{n_i,n} \to \upsilon_{n_i}$, hence we can also find K such that

$$k > K \implies \mu_{n_i,k}(X \times H \times Y) > 1 - (e/4).$$

Let $k > K$, then $v_{n_i} = \mu_{n_i,k} * v_k$. Thus

$$\begin{aligned}
1-(e/4) &< v_{n_i}(X \times H \times Y) \\
&= \int \mu_{n_i,k}\big[(X \times H \times Y)z^{-1}\big]v_k(dz) \\
&= \int_{X\times H\times Y} \mu_{n_i,k}\big[(X \times H \times Y)z^{-1}\big]v_k(dz) \\
&\qquad + \int_{X\times H^c\times Y} \mu_{n_i,k}\big[(X \times H \times Y)z^{-1}\big]v_k(dz) \\
&\leq \mu_{n_i,k}(X \times H \times Y)v_k(X \times H \times Y) + (X \times H^c \times Y) \\
&\leq \mu_{n_i,k}(X \times H \times Y)v_k(X \times H \times Y) + e/4.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mu_{n_i,k}(X \times H \times Y)v_k(X \times H \times Y)1-(e/2) \\
&\qquad \Longrightarrow \quad v_k(X \times H \times Y)1-(e/2).
\end{aligned}$$

Now for $n > k$, $v_k = \mu_{k,n} * v_n$. Then it can be shown, as above,

$$1-(e/2) < v_k(X \times H \times Y) \int \mu_{k,n}\big[(X \times H \times Y)z^{-1}\big]v_n(dz)$$

and eventually, for $k > K$ and $n > k$

$$\mu_{k,n}(X \times H \times Y) > 1-e.$$

□

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