

RESEARCH ARTICLE

Semigroups and the Generalized Road Coloring Problem

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Abstract

The road coloring problem has been open for some 25 years. This paper shows how algebraic methods, specifically semigroup theory, can be used to both generalize and shed light on the problem. Given a strongly connected digraph, the notion of a coloring semigroup is defined. The main result shows that the existence of a coloring semigroup whose kernel is a minimum rank right group of rank t implies the digraph is periodic of order t .

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1. Introduction

The road coloring problem has remained open for some 25 years. It appeared in published form for the first time in a paper by Adler, Goodwyn, and Weiss [1] and can be stated very simply: Suppose you are given a strongly connected directed graph. Is there a way to label the edges so that a *synchronizing instruction* exists? A synchronizing instruction is a sequence of edge labels such that, if followed from any of the vertices in the graph, it will take you to a given fixed vertex. A more precise definition will be provided subsequently. In [1], Adler and the others showed that aperiodicity was necessary for such an instruction to exist. The road coloring problem conjectures that aperiodicity is a sufficient condition as well.

Various partial results exist and their references are included at the end of this paper. Recently, this author together with Mukherjea [2] translated the problem into the language of semigroup theory and used the structure theory of the kernel, and some techniques from the theory of random walks on semigroups, to provide new semigroup proofs for many of the classical results. The advantage of this new perspective is one of unification of technique. An immediate consequence was an observation that the problem could be significantly generalized, and this generalization is the topic of the next section.

2. The generalized road coloring problem

A finite automaton consists of a finite set X of states, a finite set of inputs \mathcal{A} , and a state transition function $\delta: X \times \mathcal{A} \rightarrow X$. There is associated with each $a \in \mathcal{A}$ a corresponding transformation $R_a: X \rightarrow X$ defined by $R_a(x) = \delta(x, a)$, which we also write compactly as xa . The semigroup S generated under composition by $\{R_a : a \in \mathcal{A}\}$ is called the semigroup of the automaton. The state dynamics extend uniquely to a right semigroup action $(x, s) \mapsto xs: X \times S \rightarrow X$, where $xs = xa_1 \cdots a_n$ if $s = R_{a_n} \circ \cdots \circ R_{a_1}$. (See for example, Chapters 2 and 3 of [12].)

Given any automaton, one obtains in a very natural fashion a labeled digraph by taking the vertices of the graph to be the set of states X and for each $(x, a) \in X \times \mathcal{A}$ defining a directed edge from x to xa with label a . This labeled digraph determines an automaton and is a popular way of representing those of low cardinality. Note that for each vertex $v \in X$, there are $|\mathcal{A}|$ edges with initial point v , each labeled by a distinct member of \mathcal{A} . Hence X has *uniform outdegree*, i.e., all vertices of X have the same outdegree. Conversely, if one considers an unlabeled digraph $G = (V, \mathcal{E})$ of uniform outdegree d , the cardinality of \mathcal{A} , then any labeling of the edges with members of \mathcal{A} such that all the edges issuing from any given vertex have distinct labelings is called a *coloring* of the digraph. Any coloring uniquely determines an automaton $\delta: V \times \mathcal{A} \rightarrow V$, where $R_a(x) = xa$ is the terminal point of the directed edge with initial point x and label a . Thus there is a natural equivalence between finite automata and labeled digraphs of uniform outdegree, and we can freely pass between the two notions. If we are emphasizing the coloring viewpoint, then we alternatively refer to the semigroup $S = \langle \{R_a : a \in \mathcal{A}\} \rangle$ as the coloring semigroup.

We henceforth assume that digraphs have uniform outdegree.

Definition 1. Let S be a coloring semigroup of a digraph $G = (V, \mathcal{E})$. The *rank* of $s \in S$ is the cardinality $|Vs|$ of the image of s . As is well-known, the finite semigroup S has a minimal ideal or kernel, which consists of those elements of minimal rank (see [4] or [16]). This common minimal rank is called the rank of the minimal ideal.

Note that $s = a_1 \cdots a_n \in S$ is a constant map if and only if it has rank one. In the context of automata, it is known as a *synchronizing instruction*, since it takes all states to the same state in the same number of steps. In terms of colored digraphs, it means intuitively that if a traveler in a network of labeled roads gets lost, he can find his way home by following the single instruction s .

We shall be particularly interested in the case that the coloring semigroup S has a minimal ideal K that is a right group, that is, $sK = K$ for all $s \in K$. In this case it follows readily that $kS = K$ for all $k \in K$ and $sK = K$ for all $s \in S$.

The final concept required to precisely state the road coloring problem is the notion of the period of a graph.

Definition 2. Let $\pi = \{P_1, P_2, \dots, P_t\}$ be a partition of the vertices V such that if $(i, j) \in \mathcal{E}$ and $i \in P_k$, then $j \in P_{k+1}$ ($i \in P_t$ implies $j \in P_1$). Then π will be called a periodic partition of order $t \geq 2$.

Definition 3. A digraph $G = (V, \mathcal{E})$ is periodic of period $t \geq 2$ iff

$$t = \max_{\pi} [\text{card}(\pi)]$$

where this maximum is taken over all periodic partitions of G . If G has no periodic partitions, then G is aperiodic. In this case, we will refer to $\pi = \{V\}$ as the periodic partition.

The above definition can be shown to be equivalent to the more familiar one that uses the period of a vertex, where $\text{per}(v) = \text{gcd}\{|c_1|, \dots, |c_n|\}$ where c_1, \dots, c_n are the simple cycles containing v . (For one proof of this, see [5].)

The road coloring problem can now be stated in the following way.

Road coloring conjecture. Let $G = (V, \mathcal{E})$ be a d -out, strongly connected digraph. Let $\{S_i = \langle \{R_a^i : a \in \mathcal{A}\} \rangle\}$ be the set of all coloring semigroups. If G is aperiodic, then for some i , S_i contains a synchronizing instruction.

In order to generalize the road coloring conjecture to all strongly connected graphs, one needs to observe that if the completely simple kernel of a semigroup of functions is a right group, then each element in the kernel induces the same partition on the set of vertices. Here, of course, the partition π induced by a function k is the collection of subsets of V of the form $k^{-1}(v)$, for $v \in V$. In the case of the constant functions, the partition induced by the kernel is simply $\{V\}$.

Generalized road coloring conjecture. G is periodic of period $t \geq 1$ with maximal periodic partition $\pi = \{P_1, \dots, P_t\}$ iff G has a minimal (with respect to the rank of its kernel) coloring semigroup $S = \langle \{R_a : a \in \mathcal{A}\} \rangle$ whose kernel K is a right group with partition π .

Clearly if $\pi = \{P_1, \dots, P_t\}$ is the partition induced by the kernel K , then $\text{rank}(K) = t$. It will be shown in the next section that having a minimal coloring semigroup whose kernel is a right group with rank t is a sufficient condition for the graph to be periodic of order t .

3. Main result

Theorem. Let $G = (V, \mathcal{E})$ be a strongly connected d -out digraph. Suppose $S = \langle \{R_a : a \in \mathcal{A}\} \rangle$ is a minimal coloring semigroup with kernel K . If K is a right group with $\text{rank}(K) = t$, then G is periodic of order t .

Proof. For $k \in K$, $k: V \rightarrow V$, let π_k be the partition consisting of the point inverses of each point in the image of $v \mapsto vk$. For $k_1, k_2 \in K$,

$k_1 \in K = k_2S$, implies $k_1 = k_2s$ for some $s \in S$. If $vk_2 = v'k_2$, then $vk_1 = vk_2s = v'k_2s = v'k_1$. So the partition class of v in π_{k_2} is contained in that of π_{k_1} . Reversing the argument, we conclude that

$$\pi_{k_1} = \pi_{k_2} =: \pi_K.$$

Let $\pi_K = \mathcal{Q} := \{Q_1, \dots, Q_t\}$, the set whose members are the equivalence classes of π_K . Note that t is the rank of K . The right action of S on V induces a right action of S on \mathcal{Q} (if $v_1k = v_2k$ for $k \in K$, then $v_1sk = v_2sk$, since $sk \in K$ also).

Notice each member of S acts as a permutation on \mathcal{Q} . Suppose not. Then for some $v_1 \in Q_{i_1}, v_2 \in Q_{i_2}, s \in S$,

$$\begin{aligned} v_1s &\in Q_j \\ v_2s &\in Q_j, \end{aligned}$$

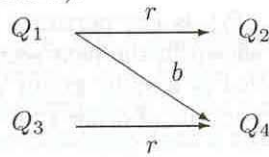
but then if $k \in K$,

$$\begin{aligned} v_1sk &\in Q_jk \\ v_2sk &\in Q_jk. \end{aligned}$$

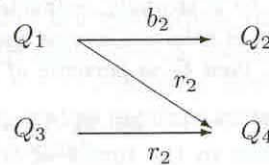
But since k is an element of the kernel, Q_jk is a single vertex. Since v_1, v_2 get mapped to the same vertex by sk (also a member of the kernel), they must have started from the same element in the partition. Thus $Q_{i_1} = Q_{i_2}$ and s is one-to-one and, therefore, a permutation on \mathcal{Q} .

Let $\pi = \{P_1, \dots, P_n\}$ be a maximal periodic partition. If $v_1, v_2 \in Q_i$, but v_1, v_2 are in different P_j , then by periodicity of the partition it follows that $v_1a_1 \cdots a_n, v_2a_1 \cdots a_n$ are in different P_j for any $s = a_1 \cdots a_n \in S$. But by definition of Q_i , $v_1s = v_2s$ for $s \in K$, a contradiction. Thus we conclude that each Q_i is contained in some P_j , i.e., the partition \mathcal{Q} refines the maximal periodic partition.

We consider two cases. The first case is that there exist two members of \mathcal{A} , say r and b , such that r and b yield different transformations on \mathcal{Q} . We must have the following (WLOG).



Now recolor in the following way.



For all other edges, $r_2 = r$, $b_2 = b$. Let S_2 be the new coloring semigroup.

Let $e = g_1 \cdot g_2 \cdot \dots \cdot g_n$ be any idempotent in $K \subset S$. Then for some $v_1 \in Q_1$, $Q_1 e = v_1$.

Form $g_1^1 \cdot g_2^1 \cdot \dots \cdot g_n^1 \in S_2$ in the following way. If $g_1 = r$, then $g_1^1 = b_2$ and if $g_1 = b$, then $g_1^1 = r_2$. Continuing in this manner, for $2 \leq i \leq n$, let

$$g_i^1 = \begin{cases} r_2 & \text{if } g_i = b \text{ and } Q_1 g_1^1 \cdot \dots \cdot g_{i-1}^1 \in Q_1 \\ b_2 & \text{if } g_i = r \text{ and } Q_1 g_1^1 \cdot \dots \cdot g_{i-1}^1 \in Q_1 \end{cases}$$

Thus, as in the second diagram above, change the color only after mapping Q_1 into Q_1 , and only if g_i is r or b . Then $Q_1 g_1^1 \cdot g_2^1 \cdot \dots \cdot g_n^1 = v_1$ and we have collapsed the set Q_1 to a single vertex.

Now $Q_2 g_1^1 \cdot g_2^1 \cdot \dots \cdot g_n^1 \in Q_j$ for some j . Then $Q_j e = v_j \in Q_j$. As above, construct $g_1^2 \cdot \dots \cdot g_n^2$ from e by recoloring it if Q_j is mapped into Q_1 . Now $Q_2 g_1^1 \cdot g_2^1 \cdot \dots \cdot g_n^1 \cdot g_1^2 \cdot \dots \cdot g_n^2 = v_j$ and we have collapsed all of Q_2 to a single vertex.

Continue in this manner for each element Q_i of the partition, each time recoloring e after mapping into Q_1 , if this occurs. Since e collapses each element in the partition to a single vertex, at the end of this process we have constructed a function $h = g_1^1 \cdot \dots \cdot g_n^1 \cdot g_1^2 \cdot \dots \cdot g_n^2 \cdot \dots \cdot g_1^t \cdot \dots \cdot g_n^t$ such that $\text{rank}(h) \leq t$.

Suppose $\text{rank}(h) = t$. Then either each vertex in the range of h is in a distinct Q class or not. If there is a Q -class with more than one vertex from the range of h , then appending e to h , once again recoloring as before, will collapse these vertices into a single vertex and reduce the rank to below t , thus contradicting minimality. If each Q -class has a single vertex in the range of h , appending first r_2 then e (appropriately recolored) to h will reduce the rank below t , once again producing a contradiction.

Now, suppose that all members of \mathcal{A} act as the same permutation σ on \mathcal{Q} . Then σ is a product of disjoint cycles. If there were more than one cycle, the graph would not be strongly connected (since one could never get out of the cycle of sets one began in). Hence σ is a cyclic permutation. It then follows that π_K is a periodic partition that refines the maximal one π . By maximality it must be equal to π . Hence $t = n$, and the theorem is proved. ■

4. A technique to complete the solution to the generalized road coloring conjecture

Using the result from the previous section, a technique suggests itself to solve the general problem. Suppose the graph is periodic of order p . Look at the collection of all minimal coloring semigroups. Suppose the kernels are of rank t . There are two possibilities. If one of the minimal kernels is a right group, from the previous result, we immediately deduce that $p = t$. If no minimal coloring semigroup has a kernel which is a right group, one must show that

some recoloring of one of the minimal semigroups produces a kernel of rank strictly less than t .

Thus, if one can prove the following conjecture, the proof of the generalized road coloring problem would be complete.

Conjecture. *Let S be a coloring semigroup whose kernel K is minimal with respect to its rank. Then K is a right group.*

Consider the following example of recoloring in the instance that a kernel is not a right group. In this example, for convenience, we use the adjacency matrix representation of a directed graph. Here A is the adjacency matrix of $G = (V, \mathcal{E})$ where $A_{ij} = 1$ whenever $(i, j) \in \mathcal{E}$. Also, notice that the transformations induced by a coloring of G have a matrix representation of their own. That is, if r is a transformation in a coloring semigroup for G , then the matrix of r is defined as $[r]_{ij} = 1$, if $ir = j$, and $[r]_{ij} = 0$ otherwise.

Example. Let G be given by the following adjacency matrix which is colored as indicated

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [r_1] + [b_1]$$

Let $S_1 = \langle r_1, b_1 \rangle$. Then S_1 can be represented by a completely simple semigroup of matrices of rank two. In fact, S_1 consists of the following eight matrices.

$$S_1 = \left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \right\}$$

Since S_1 (which is its own kernel), is not a right group, the functions within it do not all have the same partition. There are two partitions in this example.

$$\pi_1 = \{\{1, 2\}, \{3, 4\}\}, \quad \pi_2 = \{\{1, 3\}, \{2, 4\}\}$$

In general there will be many such partitions. There will always be at least one vertex v , which is paired with distinct vertices in two of these partitions. (In this example, any vertex would suffice. That is, 1 is paired with 2 in π_1 , but with 3 in π_2 .) Perhaps recoloring at such a vertex v will allow one to reduce

the rank. Computer experiments offer some hope this is true. Certainly, in this case, recoloring at 1 produces the following.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [r_2] + [b_2]$$

A simple computation will show that

$$[r_2][b_2][r_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $S_2 = \langle r_2, b_2 \rangle$ contains a rank one transformation, or equivalently, a synchronizing instruction.

5. Conclusion

In this paper, the author has shown how the structure of the kernel of a coloring semigroup continues to shed light on a longstanding problem in theoretical computer science. In addition, directed graphs, in the form of state transition diagrams, are used pervasively in both deterministic symbolic dynamics and Markov chain theory. The continuing success of semigroup theory in studying the road coloring problem offers hope that these methods could apply to open problems in these other areas, as well.

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