Convolution Products of Probability Measures on a Compact Semigroup with Applications to Random Measures

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Abstract. Given a compact semigroup S, it is well known that the set $P(S)$ of probability measures on S is itself a compact semigroup under convolution. In this paper, weak convergence of convolution products in $P(S)$ is analyzed. The conditions are then utilized to analyze the convergence of products of independent random variables taking their values in $P(S)$. Examples are constructed when S is the semigroup of $d \times d$ stochastic matrices.

1. Introduction

The study of probability on compact semigroups has a rich history [4]. Certain compact semigroups play a significant role in this history, for example, the semigroup of $d \times d$ stochastic matrices and the convolution semigroup of probability measures on a compact semigroup. These semigroups will be the focus of our current investigation.

Let (μ_n) be a sequence of probability measures on a topological semigroup, S. For a Borel set $B \subset S$, define

$$\mu_1 * \mu_2(B) = \int \mu_1(Bx^{-1})\mu_2(dx), \tag{1.1}$$

where $Bx^{-1} = \{y \in S \,|\, yx \in B\}$. We seek to determine under what conditions the convolution product

$$\mu_{k,n} = \mu_{k+1} * \mu_{k+2} * \cdots * \mu_n$$

will converge weakly, as $n \to \infty$, for all $k \geq 0$. In what follows we will denote the support of a measure μ by $\mathrm{supp}(\mu)$.

This paper intends to contibute to this program in several ways. One goal is to eventually determine verifiable necessary and sufficient conditions for convergence of these products on arbitrary finite semigroups. To accomplish this, one must first understand the behavior on finite simple semigroups, and in Section 2 some advances are made in this direction. In Section 3, we will consider conditions for convergence in $P(S)$, the set of probability measures on S, when S is a compact semigroup and applications of these conditions to random measures, defined here as the products of independent random variables taking values in $P(S)$. In this context, the work of Mindlin [6] must be mentioned. Finally, in Section 4, we will look at examples of random measures on the semigroup of $n \times n$ stochastic matrices.

2. Convolution Products on Finite Completely Simple Semigroups

In this section S will be a finite simple semigroup, i.e. a finite semigroup having no nontrivial ideals. If $E(S)$ is the set of idempotents of S, and one induces a partial order on $E(S)$ by defining $e \leq f$ iff $ef = fe = e$, it is clear that any finite semigroup will have an idempotent minimal with respect to this order. A simple semigroup with a minimal idempotent is called completely simple. Since S is completely simple, by taking a minimal idempotent e from $E(S)$ and letting $X = E(Se)$, $G = eSe$, and $Y = E(eS)$ it can be shown that the mapping $\phi_e : S \to X \times G \times Y$ defined by

$$\phi_e(s) = \big(s(ese)^{-1},\ ese,\ (ese)^{-1}s\big)$$

is an isomorphism when $X \times G \times Y$ is given the Rees product

$$(x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1, g_1(y_1x_2)g_2, y_2).$$

Notice that $y_1x_2 \in (eS)(Se) = eSe = G$. When $\phi_e(S) = X \times G \times Y$, we will write $S \cong X \times G \times Y$.

The (*sandwich*) mapping from $Y \times X$ into G, $(y, x) \mapsto yx$, plays an important role in the structure theory of these semigroups, all of which can be found in [4].

In [1], necessary and sufficient conditions were found for convergence of the convolution products when the sandwich function was trivial. The results in this section are a first attempt at generalizing this work to arbitrary sandwich functions.

Center and Mukherjea initiated the study of convolution products of non-identical measures in countable discrete groups in [2]. In that work, they made use of the notion of convergence mod a normal subgroup G, and the work that follows extends this idea to semigroups. An important reason for this investigation is the fact, demonstrated in [5], that the largest normal subgroup generated by the sandwich function plays an important role in the convergence behavior of the convolution powers.

Following [2], we will now construct a quotient structure on S and define convergence relative to this structure. Let $S \cong X \times G \times Y$ be a finite completely simple semigroup and suppose G' is a normal subgroup of G. Assume also that $YX \subset G'$. Consider the set $S/G' = \{X \times gG' \times Y : g \in G\}$. Let g', g'' be elements of G. Then

$$(X \times g'G' \times Y)(X \times g''G' \times Y) = (X \times g'G'(YX)g''G' \times Y)$$

$$= (X \times g'G'g''G' \times Y) = (X \times g'g''G' \times Y)$$

since $G'(YX) = G'$, and G' is normal. But this is an element of S/G'. In a similar manner one can show that the operation is associative. In fact, it is clear that $X \times G' \times Y$ serves as an identity for the set and given an element $X \times gG' \times Y$, $X \times g^{-1}G' \times Y$ is its inverse. Thus S/G' is a group.

Suppose (X_n) is a sequence of independent random variables taking values in S with distributions (μ_n). We will say that $X_k X_{k+1} \dots X_n$ converges mod G' with probability one iff the sequence converges in S/G' pointwise with probability one.

Theorem 2.1. *Let $(\mu_n) \subset P(S)$. Suppose $\mu_{k,n} \to v_k$ weakly for some k. Then there does not exist a normal subgroup G' of G such that $YX \subset G'$ with supp$(\mu_i) \subset X \times g_i G \times Y$ where $g_i \in G - G'$ infinitely often.*

Proof. Suppose there did exist a normal subgroup $G' \subset G$ that satisfied the above property. Then

$$\operatorname{supp}(\mu_i * \mu_{i+1}) = \operatorname{supp}(\mu_i)\operatorname{supp}(\mu_{i+1})$$

$$\subset (X \times g_i G' \times Y)(X \times g_{i+1} G' \times Y) = X \times g_i g_{i+1} G' \times Y$$

and in general,

$$\operatorname{supp}(\mu_{i,n}) \subset X \times \prod_{k=i+1}^{n} g_k G' \times Y.$$

Now assume that for some k, $\mu_{k,n} \to v_k$ as $n \to \infty$. Consider $\prod_{j=k}^{n} g_j$. This product cannot converge as $n \to \infty$, since for all j, $e \neq g_j \in G - G'$, since in a discrete group $\prod_{j=1}^{n} g_j$ converges iff there exists an N such that for all $n > N$, $g_j = e$. Hence, there exists limit points h, h', $h \neq h'$. Thus for infinitely many n, $\operatorname{supp}(\mu_{k,n}) \subset X \times hG' \times Y$, and for infinitely many n, $\operatorname{supp}(\mu_{k,n}) \subset X \times h'G' \times Y$. But if $\mu_{k,n}$ converges, this is impossible. □

The following theorem characterizes convergence mod G' for finite completely simple semigroups.

Theorem 2.2. *Let $S \cong X \times G \times Y$ be a finite completely simple semigroup. Let (X_n) be a sequence of independent random variables with values in S and with distributions $(\mu_n) \subset P(S)$. Let G' be a normal subgroup of G such that $YX \subset G'$. Then $X_1 X_2 \ldots X_n$ converges mod G' with probability one iff $\sum_{n=1}^{\infty} \mu_n\big[S - (X \times G' \times Y)\big] < \infty$.*

Proof. Let X, Y, G' be as above and suppose $\sum_{n=1}^{\infty} \mu_n\big[S - (X \times G' \times Y)\big] < \infty$. Consider $A = \{w : X_n(w) \in S - (X \times G' \times Y) \text{ i.o.}\}$. Then by the Borel-Cantelli lemma $P(A) = 0$. Thus for (P almost all) $w \in A$ there exists $N(w)$ such that $X_n(w) \in X \times G' \times Y$ for $n \geq N(w)$. Now for $k > N(w)$, $\prod_{i=1}^{k} X_i(w) = \prod_{i=1}^{N-1} X_i(w) \prod_{i=N}^{k} X_i(w)$. Since $YX \subset G'$, $\prod_{i=N}^{k} X_i(w) \subset X \times G' \times Y$. Also $\prod_{i=1}^{N-1} X_i(w) = (x, g, y) \in X \times g'G' \times Y$ for some g'

belonging to G. Hence, for all $k > N(w)$, $\prod_{i=N}^{k} X_i(w) \in (X \times g'G' \times Y)(X \times G' \times Y) = X \times g'G' \times Y \in S/G'$. Thus $X_1 X_2 \ldots X_n$ converges mod G' with probability one. Now suppose $\sum_{n=1}^{\infty} \mu_n\big[S - (X \times G' \times Y)\big] = \infty$. This implies that $P(A) = 1$ where A is defined as above. Then for (P almost all) $w \in A$, there exists (n_i) such that $X_{n_i}(w) \in S - (X \times G' \times Y)$ for all i. Thus for each i, $X_{n_i}(w) \in X \times g_{n_i} G' \times Y$ for some g_{n_i} in $G - G'$. Now $\prod_{i=1}^{n} X_i(w) \in X \times \left(\prod_{i=1}^{n} g_i G'\right) \times Y$ and since $g_i \notin G'$ i.o. and S is discrete, $\prod_{i=1}^{n} g_i$ cannot converge as $n \to \infty$. Thus $X_1 X_2 \ldots X_n$ does not converge mod G' with probability one. $\square$

3. Weak Convergence in Compact Semigroups with Applications to Random Measures

Throughout this section S will be a compact semigroup with $E(S)$ the idempotents of S. It is well known that the minimal ideal of S, known as the kernel, is completely simple. Its Rees product will, once again, be generated by choosing a minimal idempotent e. Then, if K is the kernel of S and if $X = E(Ke)$, $G = eKe$, and $Y = E(eK)$ then $K \cong X \times G \times Y$ where the product on the right is the same as in Section 2.

Notice that $P(K)$, the set of probability measures on K, is an ideal of $P(S)$, since for $\lambda \in P(K)$, $\nu \in P(S)$

$$\operatorname{supp}(\lambda * \nu) = \overline{\operatorname{supp}(\lambda)\operatorname{supp}(\nu)} \subseteq K$$

since K is closed. Since $P(K)$ is an ideal of $P(S)$, it is a subsemigroup. If we find the kernel of $P(K)$, it will be the kernel of $P(S)$ as well, since the kernel of $P(S)$ is the intersection of the closed ideals of $P(S)$.

The following theorem in [7] provides the solution, stated with the notation of this paper.

Theorem 3.1. *Let S be a compact semigroup with kernel $K = X \times G \times Y$. Let $P(S)$ be the convolution semigroup of probability measures on S. Then the kernel K' of $P(S)$ is the set of measures*

$$K' = \{\mu : \mu = \alpha \times \omega_G \times \beta\}$$

where $\alpha \in P(X)$, $\beta \in P(Y)$, and ω_G is the Haar measure on the compact group G.

Now let (X_n) be an i.i.d. sequence taking values in $P(S)$ where S is a compact semigroup and let $\hat{\mu}$ be the associated distribution. Then $\hat{\mu} \in P\big(P(S)\big)$ with $\hat{\mu}(B) = Pr(X_n \in B)$ where B is any Borel set of measures in the topology of weak convergence.

Verifiable sufficient conditions for convergence of convolution products of measures are difficult to find. The classic result of Rosenblatt [8] is less helpful in our case, since it requires the identification of the closed semigroup generated by the support of $\hat{\mu}$, $\langle\overline{\operatorname{supp}\hat{\mu}}\rangle$, which is rarely possible. In this context, the following result from [3] may be the most useful.

Theorem 3.2. *Let S be a compact semigroup with kernel K. Let $(\mu_n) \subset P(S)$ be a sequence of regular probability measures. Suppose that the following conditions hold:*

i. *There exists $x \in K$ such that for each $z \in xKx$ and any open set $N(z)$ containing z,*

$$\liminf_{n\to\infty} \mu_n\big(N(z)\big) > 0.$$

ii. *For any closed subset $C \subset \{y \in S \mid y = y^2 \text{ and } y \in xK\}$ with x as in (i), $\lim_{n\to\infty} \mu_n(S \cdot C)$ exists.*

Then for all $k \geq 0$, the sequence $(\mu_{k,n})$ converges weakly, where $\mu_{k,n} = \mu_{k+1} * \mu_{k+2} * \cdots * \mu_n$.

In (ii), C is any closed set of idempotents in xK, the minimal principal right ideal generated by the fixed x stated to exist in (i).

While the above result is true in the more general context of convolution products of measures, it is easily adapted to the current case of convolution powers, μ^n, of a single measure.

In this setting, the statement of Theorem 3.2 becomes much easier and extremely useful. Condition (ii) is trivially true and condition (i) can be written as below.

Theorem 3.2a. *Let S be a compact semigroup with kernel K. Let $\mu \in P(S)$. Suppose there exists $x \in K$ such that for each $z \in xKx$ and any open $N(z)$ containing z, $\mu\big(N(z)\big) > 0$. Then the sequence (μ^n) converges weakly as $n \to \infty$.*

Notice that for the present context the compact semigroup in question is $P(S)$ with kernel $K' = \{\mu \in P(S) : \mu = \alpha \times \omega_G \times \beta\}$ as indicated in Theorem 3.1.

Therefore we have the following result.

Theorem 3.3. *Let S be a compact semigroup and let $P(S)$ be the convolution semigroup of probability measures on S with kernel K'. Let $\hat{\mu} \in P\big(P(S)\big)$. Suppose for some $\lambda \in K'$ and any open set $N(\lambda)$ containing λ, $\hat{\mu}\big(N(\lambda)\big) > 0$. Then $(\hat{\mu}^n)$ converges weakly as $n \to \infty$.*

Proof. Note that for each $\lambda \in K'$, $\lambda * \lambda = \lambda$. But this implies that $\lambda K' \lambda = \lambda$ for each $\lambda \in K'$. Thus, from Theorem 3.2a, the result follows easily. □

4. Random measures on $n \times n$ stochastic matrices

Let S be the compact semigroup of $n \times n$ stochastic matrices under multiplication and let $P(S)$ be the convolution semigroup of probability measures on S. The kernel of K of S is the set of all rank 1 stochastic matrices, that is, stochastic matrices with identical rows.

Now it is easy to show that each matrix in K is idempotent and that, in fact, K is a right zero subsemigroup in S. In other words, if $k \in K$, then $sk = k$ for all $s \in S$. It follows that for any matrix $e = e^2 \in K$, $\phi_e(K) = \{e\} \times \{e\} \times K$. Therefore we have the following.

Theorem 4.1. *Let S be the compact semigroup of $n \times n$ stochastic matrices with kernel K, the set of all rank one stochastic matrices. Let $P(S)$ be the convolution semigroup of probability measures on S. Then the kernel of $P(S)$ is $K' = P(K)$, the set of all probability measures on K.*

Proof. Since $\phi_e(K) = \{e\} \times \{e\} \times K$ for any $e = e^2 \in K$, by Theorem **??**, $K' = \big\{\delta_e \times \delta_e \times \lambda : \lambda \in P(K)\big\}$. But this is exactly $P(K)$. □

Thus *every* probability measure on K is in the kernel K' of $P(S)$. The following examples flow from this observation.

Example 4.1. Let λ be any probability measure on rank one stochastic matrices. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be probability measures on $n \times n$ stochastic matrices. Let $\hat{\mu}(\alpha_i) = p_i$, $i = 1, \ldots, n-1$ and $\hat{\mu}(\lambda) = p_n$ where $\sum p_i = 1$, $p_i > 0$. Then by Theorem 3.3, $(\hat{\mu}^n)$ converges weakly as $n \to \infty$.

Example 4.2. In 2×2 stochastic matrices, choose $0 < p < 1$ and let

$$e = \begin{bmatrix} p & 1-p \\ p & 1-p \end{bmatrix}.$$

Let θ_n be a random variable uniformly distributed on $[-r_n, r_n]$ with $0 < r_n < \min(p, 1-p)$ for all n and $r_n \to 0$, as $n \to \infty$.

Let μ_n be the distribution of the random matrix

$$X_n = \begin{bmatrix} p+\theta_n & 1-(p+\theta_n) \\ p & 1-p \end{bmatrix} .$$

Construct $\hat{\mu} \in P\big(P(S)\big)$ so that $\hat{\mu}(\mu_n) = \frac{1}{2^n}$. Let δ_e be the delta measure on the idempotent e, and let $N(\delta_e)$ be any open set of measures containing δ_e. Then since $\mu_n \to \delta_e$, $\hat{\mu}\big(N(\delta_e)\big) > 0$. By Theorem 3.3, $(\hat{\mu}^n)$ converges weakly as $n \to \infty$.

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