Some Results Concerning Convergence of Convolution Products of Probability Measures on Discrete Semigroups

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Two types of conditions have been significant when considering the convergence of convolution products of non-identical probability measures on group and semigroups. The essential points of a sequence of measures have been useful in characterizing the supports of the limit measures. Also, enough mass eventually on an idempotent has proven sufficient for convergence in a number of structures. In this paper, both of these types of conditions are analyzed in the context of discrete non-abelian semigroups. In addition, an application to the convergence of non-homogeneous Markov chains is given.

KEY WORDS: *Convolution products, non-identical probability measures, discrete semigroups, non-homogeneous Markov chains.*

1. INTRODUCTION

Let $(\mu_n) \subset \text{Prob}(S)$ be a sequence of probability measures on a topological group or semigroup, S. Under what conditions will the convolution product

$$\mu_{k,n} = \mu_{k+1} * \mu_{k+2} \cdots * \mu_n,$$

converge weakly for all $k \geq 0$ as $n \to \infty$? As usual, for a Borel set $B \subset S$,

$$\mu_1 * \mu_2(B) = \int \mu_1(Bx^{-1})\mu_2\,(dx)$$

where $Bx^{-1} = \{y \in S | yx \in B\}$.

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This problem has been studied under various guises for some forty years. Almost all of the results were in the group context. However, a number of recent papers [1, 2, 3, 7, 10] have made progress in the semigroup setting, and this paper continues those efforts. We will work on countable discrete semigroups so topology will play no part in our discussion.

From a historical perspective, certain clear trends emerge. Two types of conditions have proven repeatedly useful in establishing convergence.

For the sequence (μ_n), define the essential points $\mathcal{E}$ as:

$$\mathcal{E} = \{x \in S \,\Big|\, \sum_{n=1}^{\infty} \mu_n(x) = \infty\}$$

and let $S' = \langle \mathcal{E} \rangle$ be the semigroup generated by the essential points. Then, if S is finite, S' is the smallest subsemigroup such that

$$\sum_{n=1}^{\infty} 1 - \mu_n(S') < \infty.$$

S' has been vital in the analysis of the finite group and finite abelian semigroup situation.

The other type of condition assumes that, eventually, mass will be placed on an idempotent or set of idempotents. Precisely, if e is the identity of a finite group, then Maximov [6] showed that

$$\liminf_{n\to\infty} \mu_n(e) > 0$$

is sufficient for convergence of the convolution products. Similarly Ruzsa [10] showed that for countable discrete abelian semigroups

$$\liminf_{n\to\infty} \mu_n(I) > 0$$

is sufficient for vague convergence, where I is the set of idempotents of S. The main body of this paper will offer a detailed analysis of both of these types of conditions.

We will use Csiszar's method of tail idempotents [4] throughout the paper. If tightness of the family of measures $\{\mu_{k,n} \,\big|\, n \geq k = 1, 2, 3, \dots\}$ is assumed, then for $(\mu_n) \subset \text{Prob}(S)$

given (n_i), there exists $(m_i) \subset (n_i)$ such that

$$\begin{aligned} &\mu_{k,m_i} \to \nu_k \\ (1) \qquad &\nu_{m_i} \to \nu_\infty = \nu_\infty^2 \\ &\nu_k = \nu_k * \nu_\infty \end{aligned}$$

Weak convergence will be denoted by an arrow. When (1) holds for some (m_i), we will write $\nu_k \equiv \nu_k(m_i)$. Now if, in addition, $\nu'_k \equiv \nu'_k(p_i)$ and $\nu_{p_i} \to \lambda$, then we have the following convolution equations, [Ref. 3 Lemma 1, p. 291]

$$\nu_k = \nu'_k * \lambda = \nu'_k * (\nu'_\infty * \lambda * \nu_\infty).$$

Notice that ν'_∞ is a right identity for ν'_k. Thus if conditions can be found such that $\nu'_\infty * \lambda * \nu_\infty = \nu'_\infty$, then we get convergence.

One final introductory remark. The algebraic condition above ($v'_\infty = \nu'_\infty * \lambda * \nu_\infty$) is exactly what is required for convergence. Suppose $\mu_{k,n} \to \nu_k$ for all $k \geq 0$. Then the set of tail limits (all limit measures of (ν_k)) forms a left zero semigroup and the condition is satisfied. Thus the following proposition is true.

Proposition 1.1. *Let S be any locally compact semigroup and $(\mu_n) \subset \mathrm{Prob}(S)$. Suppose $\nu_k = \nu_k(n_i)$, $\nu'_k = \nu_k(m_i)$ and let λ be the tail limit of (ν_k) such that $\nu_k = \nu'_k * \lambda$. Then $\mu_{k,n} \to \nu_k$ for all $k \geq 0$ iff $\nu'_\infty = \nu'_\infty * \lambda * \nu_\infty$.*

2. ESSENTIAL POINTS AND COMPLETELY SIMPLE SEMIGROUPS

For the remainder of the paper S will always be a countable discrete semigroup. If there are additional restrictions they will be stated. As indicated in the introduction, S', the semigroup (or group) generated by the essential points has provided useful information. For groups, S' is exactly the support of the unique tail idempotent of (μ_n), where the support of a measure μ will be denoted and defined by

$$\mathrm{supp}(\mu) := \{x | \mu(x) > 0\}.$$

For abelian semigroups, S' is still useful in characterizing the support of the tail idempotent. In this case, if ν_∞ is the (necessarily) unique tail idempotent then $\text{supp}(\nu_\infty)$ is the minimal ideal of S'. Unfortunately, in the non-abelian semigroup situation the problem is worse as Example 2.1 will show.

Much of this paper uses the language of completely simple semigroups. A completely simple semigroup is a semigroup with no proper ideals containing a primitive idempotent. An idempotent e is primitive if whenever $ef = fe = f$ for any other idempotent f, then $e = f$. A key fact we will use concerns the representation of a completely simple semigroup. Let $E(S)$ be the set of idempotents of S and choose $e \in E(S)$ arbitrarily. Let $X = E(Se)$, $G = eSe$, $Y = E(eS)$. Then G is a group and $X \times G \times Y$ become a semigroup when given the product

$$(x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1, g_1(y_1x_2)g_2, y_2).$$

Now define the mapping,

$$\begin{aligned} \phi_e &: S \to X \times G \times Y \\ s &\longmapsto \big(E(sSe), ese, E(eSs)\big) \end{aligned} \tag{2}$$

Then ϕ_e is an isomorphism [for a proof, see 9]. For a completely simple semigroup S, $\phi_e(S) = X \times G \times Y$ is called the Rees factorization of S. When $\phi_e(S) = X \times G \times Y$, we will often write $S \cong X \times G \times Y$.

For this example S is a completely simple semigroup with no proper right ideals (i.e. a right group). In this case, it is known that $\phi_e(S) = G \times Y$.

Example 2.1. Let S be a right group with $\phi_e(S) = G \times Y$, and let H be a finite subgroup of G. Let card $(Y) = k$. Choose $y_i \in Y$ arbitrarily and for $n \geq k$ let

$$\mu_n(e, y_j) = \frac{1}{n} \quad j \neq i$$
$$\mu_n(H \times \{y_i\}) = 1 - \left(\frac{k-1}{n}\right) \text{ distributed uniformly}$$

Then by Theorem 3 of [3], $\mu_{k,n} \to \nu_k$ and $\nu_k \to \omega_H * \delta_{y_i}$.

However, the semigroup generated by the essential points is $H \times Y$. ■

While the above example shows the essential points can generate too large a structure, at least the group part of the tail idempotent was characterized. In fact, we have the following theorem.

Theorem 2.1. *Let $S \cong X \times G \times Y$ be a countable discrete completely simple semigroup with $YX = \{e\}$, the identity of G. Let $(\mu_n) \subset \text{Prob}(S)$ and suppose $\mu_{k,n} \to \nu_k$ for all $k \geq 0$ as $n \to \infty$. Then the following are true.*

(1) $\lim_{n\to\infty} \mu_n(Sx)$ exists for all $x \in S$.

(2) There exists a finite subgroup $H \subset G$ such that if λ is a limit point of (ν_k) then $\lambda = \alpha_\lambda * \omega_H * \beta$ where ω_H is uniform on H and $\alpha_\lambda \in \text{Prob}(X)$, $\beta \in Prob(Y)$.

(3) $\sum_{n=1}^{\infty} 1 - \mu_n(X \times H \times Y) < \infty$.

Proof. Suppose $\mu_{k,n}$ converges for all $k \geq 0$. Then if (n_i) is an arbitrary subsequence

$$\mu_{k,n_i} = \mu_{k,n_i-1} * \mu_{n_i}$$

and for some subsequence $(p_i) \subset (n_i)$, $\mu_{p_i} \to \mu'$. Thus

$$\nu_k = \nu_k * \mu'$$

Now since S is discrete if $\mu_{p_i} \to \mu'$, then

$$\lim_{p_i\to\infty} \mu_{p_i}(T) = \mu'(T) \text{ for any } T \subset S.$$

Since Sx is a left ideal for all $x \in S$ and S is a right group

$$\nu_k(Sx) = \nu_k * \mu'(Sx) = \mu'(Sx)$$

Thus $\lim_{n\to\infty} \mu_n(Sx)$ exists for each x, and (1) is shown.

Part (2) of the theorem is simply restated here for convenience. Its proof is in [Ref. 3, pg. 286].

Now to show (3). Let $\mathcal{F} = \{\lambda \,|\, \nu_{n_i} \to \lambda \text{ for some } n_i\}$. Then, by (2), for all $\lambda \in \mathcal{F}$, $\text{supp}(\lambda) \cong X_\lambda \times H \times Y$, where $H \subset G$ and $Y_1 \subset Y$. Let $X \times H \times Y \cong S_1$. Since $YX = \{e\}$, S_1 is a subsemigroup of S.

Consider $S_1 z^{-1} = \{y \,|\, yz \in S_1\}$. If $(z_1, z_2, z_3) \equiv z \notin S_1$, then $z_2 \notin H$. Thus if $y \in S_1 z^{-1}$, $y \equiv (y_1, y_2, y_3)$,

$$\begin{aligned} (y_1, y_2, y_3)(z_1, z_2, z_3) &\in X \times H \times Y \\ &\Rightarrow y_2 z_2 \in H \\ &\Rightarrow y_2 \in H z_2^{-1} \\ &\Rightarrow y_2 \notin H. \end{aligned}$$

Similarly, if $(z_1, z_2, z_3) \equiv z \in S_1$, then for $(y_1, y_2, y_3) \equiv y \in S_1 z^{-1}$, it is clear that $y_2 \in H$.

Thus we have shown,

$$S_1 z^{-1} \begin{cases} \equiv X \times H \times Y & z \in S_1 \\ \subseteq X \times H^c \times Y & z \notin S_1. \end{cases}$$

Since $\mu_{k,n} \to \nu_k$ for all k and

$$\liminf_k \nu_k(X \times H \times Y) = 1$$

then given $\epsilon > 0$, there exists k, n such that

$$\mu_{k,n}(X \times H \times Y) > 1 - \epsilon.$$

So that

$$
\begin{aligned}
1-\varepsilon &< \mu_{k,n}(X\times H\times Y)\\
&= \sum_z \mu_{k,n-1}\Big[(X\times H\times Y)z^{-1}\Big]\mu_n(z)\\
&= \sum_{z\in X\times H\times Y} \mu_{k,n-1}\Big[(X\times H\times Y)z^{-1}\Big]\mu_n(z) + \sum_{z\in X\times H^c\times Y} \mu_{k,n-1}\Big[(X\times H\times Y)z^{-1}\Big]\mu_n(z)\\
&\le \mu_{k,n-1}(X\times H\times Y)\mu_n(X\times H\times Y) + \mu_{k,n-1}(X\times H^c\times Y)\mu_n(X\times H^c\times Y)\\
&= \mu_{k,n-1}(X\times H\times Y)\mu_n(X\times H\times Y) + \Big[1-\mu_{k,n-1}(X\times H\times Y)\Big]\Big[1-\mu_n(X\times H\times Y)\Big]\\
&= \mu_{k,n-1}(X\times H\times Y) - \Big[2\mu_{k,n-1}(X\times H\times Y)-1\Big]\Big[1-\mu_n(X\times H\times Y)\Big]\\
&\le \mu_{k,n-1}(X\times H\times Y) - (1-2\varepsilon)\Big[1-\mu_n(X\times H\times Y)\Big]
\end{aligned}
$$

Thus

$$\mu_{k,n+p}(X\times H\times Y) \le \mu_{k,n+p-1}(X\times H\times Y) - (1-2\varepsilon)\Big[1-\mu_{n+p}(X\times H\times Y)\Big]$$

and continuing

$$\mu_{k,n+p}(X\times H\times Y) \le \mu_{k,n}(X\times H\times Y) - (1-2\varepsilon)\sum_{m=n+1}^{n+p} 1-\mu_m(X\times H\times Y).$$

For all $p \ge 1$,

$$\sum_{m=n+i}^{n+p} 1-\mu_m(X\times H\times Y) \le \frac{1}{1-2\varepsilon}.$$

Thus

$$\sum_{n=1}^{\infty} 1-\mu_n(X\times H\times Y) < \infty. \quad \blacksquare$$

It should be noted that in part (3) of the previous theorem, the entire left and right factors of the Rees product of S were taken. Example 2.1 shows that for the right factor Y, this can not be improved. The same is true for the left factor X. To see this, under the assumption that $\mu_{k,n} \to \nu_k$ for all $k \ge 0$, notice that $\nu_k = \mu_{k+1} * \nu_{k+1}$. Then if λ_1 is any limit point of (ν_k),

$$\lambda_1 = \mu' * \lambda_2$$

for some limit points μ' of (μ_n) and λ_2 of (ν_k). Then if xS is some right ideal of S, clearly

$$\lambda_1(xS) = \mu'(xS). \tag{3}$$

Let $x \equiv (x_1, h_1, y_1) \in X \times H \times Y$ be arbitrary. Suppose $\liminf_n \mu_n(xS) = 0$, with $\mu_n(x) = \frac{1}{n}$, say. Then x is an essential point, but by equation (3), xS is not in the support of any tail limit of (μ_n). Thus the supports of the tail limits of (μ_n) can not characterize either the left or right factors of the semigroup generated by the essential points.

In the general discrete non-abelian semigroup case, it is not clear what structure, if any, should take the place of the semigroup generated by the essential points. The following theorem, however, offers a possibility.

Theorem 2.2. *Let S be a finite semigroup and $(\mu_n) \subset P(S)$. Suppose $\mu_{k,n} \to \nu_k$, for all $k \geq 0$ and let $\mathcal{F}$ be the set of tail limits of (μ_n). Let $S_1 = \bigcup_{\nu \in \mathcal{F}} \text{supp}(\nu)$ and let*

$$S_2 = \bigcap_{s \in S_1} s^{-1} S_1$$

Then S_2 is a subsemigroup of S and $\liminf_{n \to 0} \mu_n(S_2) = 1$.

Proof. Notice that $S_1 \subset S_2$ so that S_2 is not empty. Now, if $t_1, t_2 \in S_2$ and $s \in S_1$ is arbitrary then $t_1(t_2 s) = t_1 s' \in S_1$ where $t_2 s = s' \in S_1$, and hence S_2 is a subsemigroup of S.

Now suppose $\liminf_{n \to \infty} \mu_n(S_2) < 1$. Then $\limsup \mu_n(S_2^c) > \delta_1 > 0$. Since S is finite, there exists $s \in S_2^c$ such that for some subsequence (n_i), $\mu_{n_i}(s) > \delta_2 > 0$ for all i.

Since $s \in S_2^c$ there exists $s_1 \in S_1$ such that $s \notin s_1^{-1} S_1$ or $s_1 s = s_2 \notin S_1$.

It follows that

$$\mu_{k,n_i}(s_2) \geq \mu_{k,n_{i-1}}(s_1)\mu_{n_i}(s) \quad \text{for all } i$$

$$\Rightarrow \nu_k(s_2) \geq \nu_k(s_1)\delta_2$$

Now $s_1 \in S_1 \Rightarrow \nu(s_1) > 0$ for some $\nu \in \mathcal{F}$ where $\nu_{m_i} \to \nu$.

Thus

$$\nu(s_2) \geq \nu(s_1)\delta > 0 \Rightarrow s_2 \in \text{supp}(\nu) \subset S_1$$

but this is a contradiction. ■

3. THE EFFECT OF MASS ON AN IDEMPOTENT

In this section we analyze the effect of mass on an idempotent with respect to the convergence behavior of a sequence $(\mu_n) \subset \text{Prob}(S)$. For a measure $\mu \in \text{Prob}(S)$, let

$$q(\mu) = \inf_{x \in \text{supp}(\mu)} \mu(x)$$

$$\sigma(\mu) = |\,\text{supp}(\mu)|$$

Let $\nu = \mu_1 * \mu_2 \ldots \mu_n$.

Theorem 3.1. *If $q(\mu_i) \geq c > 0$ for $i = 1, \ldots, n$ and S is right cancellative, then*

$$q(\nu) \geq c^{\sigma(\nu)-1}$$

Proof. Notice for $n = 1$, $\nu = \mu$ and if $\sigma(\nu) = 1$, $c = 1$. Otherwise $q(\nu) \geq c \geq c^{\sigma(\nu)-1}$.

Let $n \geq 2$. Assume true for $n - 1$.

$$\nu(x) = \sum_{g \in \text{supp}(\mu_n)} \lambda(xg^{-1})\mu_n(g) \qquad \lambda = \mu_1 * \cdots * \mu_{n-1}$$

Case 1. $\sigma(\lambda) < \sigma(\nu)$.

Let $x \in \text{supp}(\nu)$. Then for some $g \in \text{supp}(\mu_n)$, $y \in \text{supp}(\lambda)$, $x = yg$ and

$$\nu(x) \geq \lambda(y)\mu_n(g) \geq c\lambda(y) \geq cq(\lambda)$$

Now $cq(\lambda) \geq cc^{\sigma(\lambda)-1} = c^{\sigma(\lambda)} \geq c^{\sigma(\nu)-1}$

Case 2. $\sigma(\lambda) = \sigma(\nu) = k$.

Now since $\nu = \lambda * \mu_n$,

$$\text{supp}(\nu) = \text{supp}(\lambda) \cdot \text{supp}(\mu_n)$$

$$= \bigcup_{g \in \text{supp}(\mu_n)} \text{supp}(\lambda) \cdot g$$

S is right cancellative so for each g, $|\operatorname{supp}(\lambda)\cdot g| = k$, thus they coincide.

Thus $x \in \operatorname{supp}(\lambda)\cdot g$ for all $g \in \operatorname{supp}(\mu_n)$. That is $xg^{-1} \neq \emptyset$ for each g. Therefore,

$$\nu(x) = \sum \lambda(xg^{-1})\mu_n(g) \geq \sum_g q(\lambda)\mu_n(g) = q(\lambda) \qquad \blacksquare$$

Theorem 3.2. *Let $0 < c \leq 1/2$ and suppose that $g_1, \ldots, g_n$ are such that $\mu_i(g_i) \geq c$ for all i. Let $g = g_1 \ldots g_n$. Then*

$$\nu(g) \geq c^{\sigma(\nu)-1}$$

Proof. Decompose μ_i as

$$\mu_i = \sum_{g\in\operatorname{supp}(\mu_i)} d_{ig}\,\alpha_{ig}$$

where $\alpha_{ig} = c\delta g_i + (1-c)\delta g$ so $q(\alpha_{ig}) \geq c$.

Then

$$\nu = \sum_{x_i\in\operatorname{supp}(\mu_i)} d_{1x_1}\ldots d_{nx_n}\alpha_{1x_1}\ldots\alpha_{nx_n}$$

Now by Th. 3.1 $q(\alpha_{1x_1}\cdots\alpha_{nx_n}) \geq\ c^{\sigma(\alpha 1x_1\ldots\alpha nx_n)-1} \geq c^{\sigma(\nu)-1}$

Thus

$$\begin{aligned}\nu(g) &= \sum_{x_i\in\operatorname{supp}(\mu_i)} d_{1x_1}\cdots d_{nx_n}\alpha_{1x_1}\cdots\alpha_{nx_n}(g)\\ &\geq (c^{\sigma(v)-1})\sum d_{1x_1}\ldots d_{nx_n}\\ &\geq c^{\sigma(v)-1} \quad \blacksquare\end{aligned}$$

Now let S be completely simple, and let $\mu_i \in \operatorname{Prob}(S)$ be such that $\mu_i = c\delta a_{i1}+(1-c)\delta a_{i2}$ for some a_{i1}, a_{i2}. Fix n, and write $\lambda = \mu_1 * \mu_2 \cdots * \mu_{n-1},\ \mu = \mu_n,\ \nu = \lambda * \mu = \mu_1 * \mu_2 \cdots * \mu_n$.

Also set $a_{n1} = a_1,\ a_{n2} = a_2,\ a_{n-1,1} = b_1\ a_{n-1,2} = b_2$

Define

$$\sigma_i(\nu) = |\operatorname{supp}(\nu) \cap Sa_i| \quad i = 1, 2$$
$$\overline{\sigma}(\nu) = \tfrac{1}{2}\Big(\sigma_1(\nu) + \sigma_2(\nu)\Big)$$
$$q_i(\nu) = \min\Big\{\nu(x)\Big| x \in \operatorname{supp}(\nu) \cap Sa_i\Big\}$$
$$\overline{q}(\nu) = \begin{cases} q(\nu) = q_1(\nu) = q_2(\nu) & \text{if} \quad Sa_1 = Sa_2 \\ \min \frac{q_1(\nu)}{c}, \frac{q_2(\nu)}{1-c} & \text{if} \quad Sa_1 \neq Sa_2 \end{cases}$$

Theorem 3.3. $\overline{q}(\nu) \geq c^{4(\overline{\sigma}(\nu)-1)}$.

Lemma. *$\overline{\sigma}(\nu) \geq \overline{\sigma}(\lambda)$ and if equality holds then*

(i) $k = \overline{\sigma}(\nu) = \overline{\sigma}(\lambda)$ is an integer

(ii) all nonempty sets of the form $\operatorname{supp}(\lambda) \cap Sx$ or $\operatorname{supp}(\nu) \cap Sx$ have k elements

(iii) for any $i, j \in \{1, 2\}$

$$(\operatorname{supp}(\lambda) \cap Sa_i)a_j = \operatorname{supp}(\nu) \cap Sa_j$$

(iv) $\overline{q}(\nu) \geq \overline{q}(\lambda)$

Proof. Let

$$B_i = \operatorname{supp}(\lambda) \cap Sb_i$$

$$A_i = \operatorname{supp}(\nu) \cap Sa_i$$

Clearly, for $j = 1, 2$, $A_i \subset B_j a_i$ thus

$$|A_i| \geq \max |B_j|$$

and hence,

$$\begin{aligned} \overline{\sigma}(\nu) &= \frac{|A_1| + |A_2|}{2} \\ &\geq \max |B_j| \\ &\geq \frac{|B_1| + |B_2|}{2} \\ &= \overline{\sigma}(\lambda) \end{aligned}$$

Now suppose $\overline{\sigma}(\nu) = \overline{\sigma}(\lambda)$. Then $|B_1| = |B_2| = k = |A_1| = |A_2|$ which shows (i) and (ii).

Also every inclusion $A_i \supset B_j a_i$ must be equality which shows (iii).

To prove (iv), we distinguish four cases

Case 1. $Sb_1 = Sb_2$, $Sa_1 = Sa_2$.

Then $\overline{q}(\nu) = q(\nu) = \min_{x \in \text{supp}(\nu)} \nu(x) = \nu(\hat{x})$ for some $\hat{x}$ and

$$\nu(\hat{x}) = \sum_g \lambda(xg^{-1})\mu(g) = c\lambda(xa_1^{-1}) + (1-c)\lambda(xa_2^{-1})$$

Now $B_1 a_i = A_i$, but, $A_1 = A_2$, so $B_1 a_1 = B_1 a_2 = A$. For $\hat{x}$, notice that both $\hat{x}a_1^{-1}$, $\hat{x}a_2^{-1}$ are nonempty, singleton sets in $\text{supp}(\lambda)$.

$$\overline{q}(\nu) = \nu(\hat{x}) \geq c\overline{q}(\lambda) + (1-c)\overline{q}(\lambda) = \overline{q}(\lambda)$$

Case 2. $Sb_1 \neq Sb_2$, $Sa_1 = Sa_2$.

Again let $\nu(\hat{x}) = \overline{q}(\nu) = c\lambda(\hat{x}a_1^{-1}) + (1-c)\lambda(\hat{x}a_2^{-1})$

Since $B_1 a_1 = B_2 a_1 = A = B_1 a_2 = B_2 a_2$, then $\hat{x} \in \text{supp}(\nu) = A$ implies that $\hat{x}a_1^{-1}$ intersects both B_1 and B_2, so

$$\lambda(xa_1^{-1}) \geq q_1(\lambda) + q_2(\lambda) \geq \overline{q}(\lambda)$$

since $q_1(\lambda) \geq c\overline{q}(\lambda)$, $q_2(\lambda) = (1-c)\overline{q}(\lambda)$. Thus $\overline{q}(\nu) \geq c\overline{q}(\lambda) + (1-c)\overline{q}(\lambda) = \overline{q}(\lambda)$

Case 3. $Sb_1 = Sb_2$, $Sa_1 \neq Sa_2$.

For $\hat{x} \in \text{supp}(\nu)$, $\nu(\hat{x}) = q_i(\nu)$

Thus,

$$\begin{aligned} v(\hat{x}) &= c_i \lambda(\hat{x}a_i^{-1}) \\ &\geq c_i q(\lambda) = c_i \overline{q}(\lambda) \end{aligned}$$

where $c_1 = c$, $c_2 = 1 - c$.

Thus $\overline{q}(\nu) = \min \frac{q_2(\nu)}{c} \geq \overline{q}(\lambda)$

Case 4. $Sb_1 \neq Sb_2$, $Sa_1 \neq Sa_2$.

Again, let $\nu(\hat{x}) = q_i(\nu)$ for fixed but arbitrary $i \in \{1, 2\}$.

Now if $\hat{x} \in Sa_i$ then $\hat{x} \in B_1 a_i$ and $\hat{x} \in B_2 a_i$

Thus

$$\begin{aligned} q_i(\nu) = \nu(\hat{x}) &= c_i \lambda(\hat{x} a_i^{-1}), \quad c_i \text{ as above.} \\ &\geq c_i(q_1(\lambda) + q_2(\lambda)) \\ &\geq c_i \overline{q}(\lambda) \end{aligned}$$

which implies that $\overline{q}(\nu) \geq \overline{q}(\lambda)$. ■

Proof of Theorem 3.3. We proceed by induction on the order of the convolution. The case $n = 1$ is clear.

Suppose $\overline{\sigma}(\nu) > \overline{\sigma}(\lambda)$. Then $\overline{\sigma}(\nu) - \overline{\sigma}(\lambda) \geq \frac{1}{2}$.

Now for $x \in \operatorname{supp}(\nu)$ we have for some $u \in \operatorname{supp}(\lambda)$, $g \in \operatorname{supp}(\mu)$,

$$\begin{aligned} \nu(x) &\geq \lambda(u)\mu(g) \\ &\geq c\lambda(u) \geq cq(\lambda) \geq c^2 \overline{q}(\lambda) \end{aligned}$$

but by assumption

$$\begin{aligned} c^2 \overline{q}(\lambda) &\geq c^2 c^{4[\overline{\sigma}(\lambda) - 1]} \\ &\geq c^2 c^{4[\overline{\sigma}(\nu) - \frac{3}{2}]} \\ &= c^{4[\overline{\sigma}(\nu) - 1]} \end{aligned}$$

Part (IV) of the lemma provides the information needed in the case $\overline{\sigma}(\nu) = \overline{\sigma}(\lambda)$. ■

Corollary 3.1. $q(\nu) \geq c\overline{q}(\nu) \geq c^{4\overline{\sigma}(\nu) - 3} \geq c^{4\sigma(\nu) - 3}$.

The following theorem now can be deduced from the above corollary in the same manner as Theorem 3.2 was generated from Theorem 3.1.

Theorem 3.4. *Let S be a completely simple semigroup and $(g_i) \subset S$. Suppose for all $i \geq 1$, $\mu_i(g_i) \geq c$, for some c, $0 < c \leq \frac{1}{2}$. Then if $\nu = \mu_1 * \mu_2 * \cdots * \mu_n$,*

$$\nu(g_1 g_2 \ldots g_n) \geq c^{4\sigma(\nu)-3}$$

Corollary 3.2. *Suppose S is a finite completely simple semigroup. Let $e = e^2 \in S$. Suppose for $(\mu_n) \subset \mathrm{Prob}(S)$, $\liminf_{n\to\infty} \mu_n(e) > 0$. Let $\nu_k \equiv \nu_k(n_i)$. Then $e \in \mathrm{supp}(\lambda)$, where λ is an arbitrary limit point of (ν_k).*

Proof. Since $\liminf_{n\to\infty} \mu_n(e) > 0$, there is a $c > 0$ such that $\mu_k(e) \geq c$ for all $k \geq k_0$. But by Theorem 3.4, $\mu_{k,n}(e) \geq c^{4|S|-3} = \hat{c} > 0$ for all $n > k \geq k_0$. Thus if $\nu_k \equiv \nu_k(n_i)$, $\nu_k(e) \geq \liminf_{n\to\infty} \mu_{k,n}(e) \geq \hat{c}$. But then $\liminf_{k\to\infty} \nu_k(e) \geq \hat{c} > 0$ and the corollary follows. ■

Theorem 3.5. *Let S be a finite completely simple semigroup and $(\mu_n) \subset P(S)$. Suppose that $\lim_{n\to\infty} \mu_n(Sx)$ exists for all $x \in S$, and that for some idempotent $e \in S$,*

$$\liminf_{n\to\infty} \mu_n(e) > 0.$$

Then $\mu_{k,n} \to \nu_k$ as $n \to \infty$ for all $k \geq 0$.

Proof. Suppose that $\liminf_{n\to\infty} \mu_n(e) > 0$ for some $e = e^2 \in S$. Then if λ is an arbitrary tail limit measure of (μ_n), by Corollary 3.2, $e \in \mathrm{supp}(\lambda)$.

Let $\nu_k = \nu_k(n_i)$ and $v'_k = v'_k(m_i)$ with $\nu_{n_i} \to \nu_\infty = \nu_\infty^2$ and $v'_{m_i} \to \nu'_\infty = \nu'^2_\infty$. Now for some subsequence of (m_i) (still refering to it as (m_i) for convenience), $\nu_{m_i} \to \lambda$. Also, as indicated earlier,

$$\begin{aligned} \nu_k &= \nu'_k * \lambda \\ &= \nu'_k * (\nu'_\infty * \lambda * \nu_\infty) \end{aligned}$$

Now, we can use e to find the Rees factorization of S and define ϕ_e as in (2) above. Since e is in the support of every tail idempotent of (μ_n), by Corollary 7 in [1],

$$\phi_e\big[\mathrm{supp}(\nu_\infty)\big] = X_1 \times H_1 \times Y_1$$

$$\phi_e\big[\mathrm{supp}(\nu'_\infty)\big] = X_2 \times H_2 \times Y_2 \tag{4}$$

and

$$\phi_e\big[\text{supp}(\lambda)\big] = X_2 \times hH_1 \times Y_1 = X_2 \times H_2 h \times Y_1$$

for some $h \in e \cdot \text{supp}(\lambda) \cdot e$.

Now since

$$\mu_{k,n_i} = \mu_{k,n_i-1} * \mu_{n_i}$$

then $\nu_k = \beta_k * \mu'$ where μ' is some limit point of (μ_{n_i}).

Notice for any $x \in S$, Sx is a left ideal of S. Thus, since S is completely simple, for all $k \geq 0$,

$$\begin{aligned}\nu_k(Sx) &= \sum_{y \in S} \beta_k\left[(Sx)y^{-1}\right]\mu'(y)\\ &= \mu'(Sx).\end{aligned}$$

But since $\lim_{n\to\infty} \mu_n(Sx)$ exists for each $x \in S$, then for arbitrary limit points μ', $\mu^{''}$ of (μ_n)

$$\mu'(Sx) = \mu^{''}(Sx).$$

Thus, similarly, $\mu_{k,m_i} = \mu_{k,m_i-1} * \mu_{m_i}$ implies that for all $k \geq 0$,

$$\nu'_k(Sx) = \mu^{''}(Sx) = \mu'(Sx) = \nu_k(Sx).$$

But this shows that

$$\nu_\infty(Sx) = \lambda(Sx) = \nu'_\infty(Sx). \tag{5}$$

Now ϕ_e is an isomorphism, which shows that for $x \in S$,

$$\phi_e(Sx) = X \times G \times \{y_i\}$$

where $x \equiv (x_i, g_i, y_i)$. But (5) above indicates that each minimal left ideal with non-zero mass for one tail limit measure, has non-zero mass for every tail limit measure. In particular, $Y_1 = Y_2 = \hat{Y}$ in (4) above.

Notice that e is the identity for the group $G = eSe$ and $e \equiv (e, e, e)$.

However, since $e \in \text{supp}(\lambda)$,

$$(e, e, e) \in \phi_e\big[\text{supp}(\lambda)\big] = X_2 \times hH_1 \times Y = X_2 \times H_2 h \times Y_1$$

which easily shows that $H_1 = hH_1 = H_2 h = H_2 = H$. Therefore the supports of the limit measures ν_∞, ν'_∞ and λ are as follows.

$$\begin{aligned}
\phi_e\big[\text{supp}(\nu_\infty)\big] &= X_1 \times H \times \hat{Y}, \quad \hat{Y} X_1 \subset H \\
\phi_e\big[\text{supp}(\nu'_\infty)\big] &= X_2 \times H \times \hat{Y}, \quad \hat{Y} X_2 \subset \hat{H} \\
\phi_e\big[\text{supp}(\lambda)\big] &= X_2 \times H \times \hat{Y}
\end{aligned}$$

Now in [3] it was shown that the idempotent measures ν_∞, ν'_∞ themselves have a product structure. In fact,

$$\begin{aligned}
\nu_\infty &= \lambda * w_H * \lambda_2; \quad \text{supp}(\lambda_1) = X_1, \ \text{supp}(\lambda_2) = \hat{Y} \\
\nu'_\infty &= \lambda'_1 * w_H * \lambda_2; \quad \text{supp}(\lambda'_1) = X_2
\end{aligned}$$

where the third factors of ν_∞, ν'_∞ are equal from (4) above.

Then

$$\begin{aligned}
\nu'_\infty * \lambda * \nu_\infty &= \lambda'_1 * w_H * (\lambda_2 * \lambda * \lambda_1) * w_H * \lambda_2 \\
&= \lambda'_1 * w_H * \lambda_2 = \nu'_\infty
\end{aligned}$$

since $\text{supp}(\lambda_2 * \lambda * \lambda_1) \subset H$.

We use this fact along with Proposition 1.1 to deduce that $\mu_{k,n} \to \nu_k$ for all $k \geq 0$ as $n \to \infty$. ■

4. An application to non-homogeneous Markov chains.

One of the most important applications of this theory is to the convergence of non-homogeneous Markov chains. Let (X_k) be a non-homogeneous Markov chain with state space $\{1, \ldots, n\}$ and $n \times n$ transition matrices (T_k). Then (T_k) is often referred to as a non-homogeneous stochastic chain. No assumptions will be made concerning the irreducibility of the transition matrices. One question concerns conditions under which the products

$$T_{k,n} = T_{k+1} T_{k+2} \ldots \cdot T_n$$

will converge for all k. Clearly, conditions which are sufficient for convergence of the products will ensure, independent of initial distribution, convergence in distribution of (X_k). The most useful conditions are those based only on the individual matrices, not on products of unspecified length. To get conditions of this type we will exploit the relationship between probability measures on binary stochastic matrices and stochastic matrices [see Section 3 of Ref. 8].

Example 4.1.

Let (T_k) be a sequence of 3×3 stochastic matrices such that for each k,

$$T_k = \begin{bmatrix} p_1^k & p_2^k & p_3^k + p_4^k \\ p_1^k & p_2^k & p_3^k + p_4^k \\ p_3^k & p_4^k & p_1^k + p_2^k \end{bmatrix}$$

Then

$$T_k = \sum_{i=1}^{4} p_i^k \, B_i$$

where

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Notice that $S = \{B_1, B_2, B_3, B_4\}$ is a completely simple semigroup of matrices and that B_1 and B_2 are its idempotents. Also,

$$SB_1 = SB_3 = \{B_1, B_3\}$$

$$SB_2 = SB_4 = \{B_2, B_4\}.$$

Now define $\mu_k(B_i) = p_i^k$. Then by Theorem 3.5, $\mu_{k,n}$ will converge (and therefore $T_{k,n}$) if either $\lim_k \inf p_1^k > 0$ or $\lim_k \inf p_2^k > 0$, and if both

$$\lim_{k\to\infty} p_1^k + p_3^k \text{ and } \lim_{k\to\infty} p_2^k + p_4^k$$

exist.

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