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RESEARCH ARTICLE

Completely Simple Semigroups, Lie Algebras, and the Road Coloring Problem

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Abstract

Consider a semigroup generated by matrices associated with an edge-coloring of a strongly connected, aperiodic digraph. We call the semigroup *Lie-solvable* if the Lie algebra generated by its elements is solvable. We show that if the semigroup is Lie-solvable then its kernel is a right group. Next, we study the Lie algebra generated by the kernel. Lie algebras generated by two idempotents are analyzed in detail. We find that these have homomorphic images that are generalized quaternion algebras. We show that if the kernel is not a direct product, then the Lie algebra generated by the kernel is not solvable by describing the structure of these algebras. Finally, we discuss an infinite class of examples that are shown to always produce strongly connected aperiodic digraphs having kernels that are not right groups.

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1. Introduction

Studying the dynamics inherent in state-transition diagrams, either in the stochastic setting as Markov chains or in deterministic models as automata, continues to be a rich source of interesting mathematics and applications. Even for finite state systems, fascinating unsolved problems exist. *Ĉerný's conjecture* concerning the length of a synchronizing instruction for deterministic automata is one such example. Another open problem in this area is referred to as the "road coloring problem" and was the initial motivation for this work.

The road coloring problem appeared in published form for the first time in a paper by Adler, Goodwyn, and Weiss [1] and can be stated very simply: Suppose you are given a strongly connected directed graph (or state-transition diagram). Is there a way to label the edges so that a synchronizing instruction exists? A precise definition of synchronizing instruction will be provided below. In [1], it was shown that aperiodicity was necessary for such an instruction to exist. The road coloring problem conjectures that aperiodicity is a sufficient condition as well.

1.1. Background

A finite automaton consists of a finite set V of states, a finite set of inputs A, and a state transition function $\delta\colon V\times A\to V$. We can associate with each $a\in A$

a corresponding transformation $R_a\colon V \to V$ defined by $R_a(v) = \delta(v,a)$, which we write compactly as va. The semigroup S generated under composition by $\{R_a\colon a \in A\}$ is called the semigroup of the automaton. Given any automaton, one obtains a labeled digraph by taking the vertices of the graph to be the set of states V and for each $(v,a) \in V \times A$ defining a directed edge from v to va with label a. Note that the constructed directed graph has *uniform outdegree*, i.e., each vertex of V has the same number of edges leading from it.

If one considers an unlabeled digraph $G = (V, \mathcal{E})$ of uniform outdegree d, then any labelling of the edges with members of A, where $|A| = d$, such that each edge issuing from any given vertex has a distinct label is called a *coloring* of the digraph. Any such coloring uniquely determines an automaton $\delta\colon V \times A \to V$, where $R_a(v) = va$ is the terminal point of the directed edge with initial point v and label a. We henceforth assume that digraphs have uniform outdegree, identifying the coloring with the set $\mathcal{C} = \{R_a\colon a \in A\}$, and we refer to the semigroup $S = \langle \mathcal{C} \rangle$, generated by $\mathcal{C}$, as the *coloring semigroup* for the given labelling.

Definition 1. Let S be a coloring semigroup of a digraph $G = (V, \mathcal{E})$. The *rank* of $s \in S$ is the cardinality $|Vs|$ of the image of s. Any finite transformation semigroup S has a minimal ideal or *kernel*, which consists of the elements of minimal rank (see [5]). This common minimal rank is called the *rank of the kernel*.

A *synchronizing instruction* for an edge-labelled digraph is thus any transformation of rank one. The road coloring problem can now be stated in the following way.

Road Coloring Conjecture. *Let $G = (V, \mathcal{E})$ be a d-out, strongly connected digraph. Let $\{S_i\}$ be the set of all coloring semigroups. If G is aperiodic, then for some i, S_i contains a synchronizing instruction.*

The notion of period used here is the standard one from Markov chain theory, where for any vertex v in a directed graph, $\operatorname{per}(v) = \gcd\{|c_1|, \ldots, |c_n|\}$ where $c_1, \ldots, c_n$ are the simple cycles containing v. The assumption of strong connectivity ensures that all vertices have the same period.

Notation. We use rank and trace to denote the rank and trace respectively of a matrix. For a function, rank and trace refer to its representing matrix.

1.2. Semigroups and kernels

Let S be a coloring semigroup of a strongly connected digraph with kernel K. Let $E(S)$ be the set of idempotents of S. Impose on $E(S)$ the order $e \le f$ iff $ef = fe = e$. Form the Rees product $X \times G \times Y$ of K in the usual manner choosing an idempotent, e_0, minimal with respect to this order and letting

$$X = E(Ke_0), \quad G = e_0 K e_0, \quad Y = E(e_0 K)$$

with product

$$(x_1, g_1, y_1)(x_2, g_2, y_2) := (x_1, g_1(y_1x_2)g_2, y_2)$$

Notice that $y_1x_2 \in G$. The sandwich function

$$\phi\colon\ Y \times X \to G, \quad \phi(y, x) = yx$$

plays an important rôle in this paper. In particular, recall that if $\phi(y, x) = e_0$ for all $(y, x) \in Y \times X$, then $X \times G \times Y$ is a direct product.

Recall that if K is the kernel of a finite transformation semigroup S, the sets X, G, and Y can be further characterized. Suppose S is a semigroup of functions on the set $\{1, \ldots, n\}$. Then each idempotent $e \in X$ can be associated with a partition π_e of the set $\{1, \ldots, n\}$ in the standard way: $\pi_e = \{e^{-1}(j)\colon 1 \leq j \leq n\}$. Thus the set X can also be thought of as a collection of partitions of the set $\{1, \ldots, n\}$. Similarly each idempotent $f \in Y$ can be associated with its range $R_f = \{jf\colon 1 \leq j \leq n\}$, where we employ the convention of writing inputs to the transformation on the left. The group $G = e_0Ke_0$ can then be seen as the subgroup of transformations within K having the same partition and range as e_0.

The ideal structure of K will also play a rôle in our discussions. Each set of the form eK with $e \in X$ is a minimal right ideal consisting of the set of all transformations in K having partition π_e. Similarly, Kf with $f \in Y$ is a minimal left ideal consisting of the set of all transformations in K having range R_f. When $X = E(Ke_0) = \{e_0\}$, a single idempotent, K is, of course, a right group. Notice that when a coloring semigroup has a synchronizing instruction or rank one instruction, K is easily shown to be a right group.

The importance of right groups in the context of the road coloring problem has been shown in recent papers [2, 3] and this paper continues an exploration of their rôle. In fact, we have the following theorem.

Theorem 1. *Suppose a strongly connected, aperiodic digraph G has a coloring semigroup S whose kernel K is a right group. Then G has a coloring semigroup that contains a synchronizing instruction.*

Proof. Suppose the (unique) vertex partition induced by the right group K, is $\Pi = \{P_1, P_2 \ldots P_t\}$, where $\operatorname{rank}(K) = t \geq 2$ (as we are done if $t = 1$).

Notice that every element of S, and therefore the generators of S, act as permutations on the elements of Π. Suppose not. Then there are vertices $v_i \in P_i$, $v_j \in P_j$, and an element s in S such that

$$v_is \in P_m, \qquad v_js \in P_m$$

Then, if k is any element of K,

$$v_isk \in P_mk, \qquad v_jsk \in P_mk$$

But $P_mk = w$, since the elements of the kernel collapse the partition elements to a single vertex. Since sk is in the kernel, the $\{v\colon vsk = w\}$ is one of the partition elements. Thus $P_i = P_j$, showing that s is a permutation on partition elements.

Form the quotient graph induced by the coloring on the "vertices" $\Pi = \{P_1, P_2, \ldots P_t\}$. Since the generators act as permutations on the elements of Π, the resulting graph will be "Eulerian", having the indegree of each vertex equal to its outdegree. Now, we will use the algorithm introduced by Kari [7] to find a coloring that synchronizes this Eulerian graph. This is done in the following way.

Choose any element $w_0 = c_1 c_2 \cdots c_l$ from the original kernel K, each c_i being a generator of S. Note that $P_1 w_0$ is a single vertex in the original graph. Proceeding as in [7], choose a vertex P_{i_0}, in the quotient graph, such that for colors r and b,

$$P_{i_0} r \neq P_{i_0} b$$

For step one, swap the labels of the r-edges leaving P_{i_0} with the b-edges leaving P_{i_0}. No other edge labels are changed.

For step two, we reassign colors in w_0 as follows. If $P_1 = P_{i_0}$ then change c_1. If c_1 is either r or b, swap r with b, otherwise leave it unchanged. Working through w_0, whenever $P_1 c_1 c_2 \cdots c_{j-1} \subset P_{i_0}$, change c_j accordingly. This results in a new word, $w_1 = c_1' c_2' \cdots c_l'$, with $P_1 w_1$ remaining a single vertex.

Each stage of this procedure is performed in the same way. At stage k, Kari's algorithm guarantees that the new quotient graph is Eulerian with fewer vertices. Step one consists of finding a vertex in this new quotient graph and relabelling as described above. For step two, the word w_{k-1} is changed each time the vertices of P_1 map to that partition element that was changed by this kth relabelling, ensuring that $P_1 w_k$ is a single vertex.

This procedure terminates in a synchronizing instruction w' for the quotient graph induced by the kernel, i.e., w' maps every P_j to a single P_{j_0}. Since the graph is strongly connected at the quotient level, there exists an instruction w'' such that $P_{j_0} w'' \subset P_1$. If the procedure requires m steps then $w_{\text{sync}} = w' w'' w_m$ is a synchronizing instruction for the original graph. ■

This work introduces the Lie algebra generated by the coloring transformations. We give a sufficient condition in terms of the structure of the Lie algebra implying that the kernel is a right group. In addition, we analyze the underlying structure of the Lie algebra generated by the kernel.

2. Solvability and right groups

Throughout this work, we consider Lie algebras over $\mathbb{C}$. In general, if $\mathfrak{g}$ is a Lie algebra, its *derived algebra,* $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, is the linear span of all products $[x, y]$, $x, y \in \mathfrak{g}$. This is an ideal of $\mathfrak{g}$. Continuing, one defines a sequence of ideals

$$\mathfrak{g} \supset \mathfrak{g}' \supset \mathfrak{g}'' \supset \cdots \supset \mathfrak{g}^{(n)} \supset \cdots$$

with $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$, which is the *derived series* of $\mathfrak{g}$. If this series terminates at 0 after finitely many steps, then $\mathfrak{g}$ is *solvable*. Note that the last nonzero term of the series of a solvable algebra is abelian.

The main property of a solvable Lie algebra that we are using here is *Lie's Theorem* to the effect that (over an algebraically closed field) a solvable Lie algebra of matrices can be simultaneously upper-triangularized ([6, p. 50], also see [8, p. 21–26]). In this context the work by Radjavi & Rosenthal [8] is of particular interest. Especially, Radjavi's Theorem on permutable traces [8, p. 33] is part of the inspiration behind the proof.

There are three Lie algebras associated with a coloring $\mathcal{C}$ of a strongly connected digraph with coloring semigroup S and its kernel K. In general, we denote by $\mathfrak{g}(\cdot)$ the Lie algebra generated by transformations from a given set. In this context, we specify $\mathcal{S} = \mathfrak{g}(S)$ the Lie algebra generated by the full semigroup. The underlying set is the same as the semigroup algebra (without 1) with commutator as product. The Lie algebra $\mathcal{L} = \mathfrak{g}(\mathcal{C})$, generated by the transformations in $\mathcal{C}$, is a subalgebra of $\mathcal{S}$ and is our main object of interest. Finally, $\mathcal{K} = \mathfrak{g}(K)$, is a subalgebra of $\mathcal{S}$, that is, in fact, a Lie ideal of $\mathcal{S}$ since K is an ideal in the semigroup. The main feature is that all three Lie algebras are solvable if and only if the generators $\mathcal{C}$ are simultaneously triangularizable if and only if $\mathcal{L}$ is solvable. In that case, we call the graph *Lie-solvable.*

Theorem 2. *If $\mathcal{L}$ is solvable, then the kernel K is a right group.*

We start with a lemma.

Lemma 1. *If $\mathcal{L}$ is solvable, then the kernel K is isomorphic to a Rees product semigroup that is a direct product.*

Proof. We want to show that

$$K \cong X \times G \times Y$$

where if $(x_1, g_1, y_1), (x_2, g_2, y_2) \in X \times G \times Y$ then

$$(x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1 x_2, g_1 g_2, y_1 y_2).$$

It is well known that this is equivalent to the property:

$$\text{for any two idempotents } e, f \in K,\ (ef)^2 = ef.$$

In other words, the idempotents of K should form a *band,* a semigroup of idempotents.

Note that if we show that $\operatorname{rank}(ef) = \operatorname{trace}(ef)$, then we are done, since for a function

$$\operatorname{rank}(f) = \operatorname{trace}(f) \text{ iff } f \text{ is an idempotent.}$$

Let $e, f \in K$, then, since ef is contained in some maximal subgroup of K, $(ef)^p$ is an idempotent for some $p \geq 1$. Since $\mathcal{L}$ is solvable, the matrices of $\mathcal{L}$

can be made simultaneously upper triangular, thus $\operatorname{trace}((ef)^p) = \operatorname{trace}(e^p f^p)$. Also, since both $(ef)^p$ and ef are in K, $\operatorname{rank}((ef)^p) = \operatorname{rank}(ef)$. Thus we have $\operatorname{rank}(ef) = \operatorname{rank}((ef)^p) = \operatorname{trace}((ef)^p) = \operatorname{trace}(e^p f^p) = \operatorname{trace}(ef)$, since $e^p = e$ and $f^p = f$. Therefore, ef is an idempotent and it follows that the Rees product is a direct product. ■

Now we proceed with the proof of the theorem.

Proof. Consider two partitions π_1, π_2 of the vertices V, corresponding to minimal right ideals of K. Let $\pi_1 = \{P_1, P_2, \ldots, P_r\}$, $\pi_2 = \{Q_1, Q_2, \ldots, Q_r\}$, and suppose $R_1, R_2, \ldots, R_t$ are all possible ranges of the functions in K. (Note: r is the rank of the kernel.)

Label the idempotents $e_i = E\big(H(\pi_1, R_i)\big)$ and $f_i = E\big(H(\pi_2, R_i)\big)$ where $H(\pi_j, R_i)$ is the H-class determined by partition π_j and range R_i. Notice that since K is a direct product, $e_i f_j = e_j$, since $e_i f_j \in H(\pi_1, R_j)$ and is an idempotent.

Let $R_1 = \{x_1, x_2, \ldots, x_r\}$, with $x_i \in P_i$. Consider the sets $O_j = \{x_j e_i\colon\ i = 1, \ldots, t\}$. Each idempotent acts as a set identity function on its range, so that $x_j e_i \in P_j$ for $i = 1, \ldots, t$ and $j = 1, \ldots, r$. Hence $O_j \subset P_j$ for $j = 1, \ldots, r$. In particular, $O_{j_1} \cap O_{j_2} = \emptyset$, if $j_1 \neq j_2$.

Notice that if, for some $x \in P_j$, $xe_i = x$, then $x_j e_i = x$ will pick up $x \in O_j$. Thus,

$$V = \bigcup_{j=1}^{r} O_j$$

since $\bigcup_{i=1}^{t} R_i = V$ so that every element of V is in the range of some idempotent e_i. So, in fact, $O_j = P_j$ for $j = 1, \ldots, r$.

Similarly, $Q_j = \{x_j f_k\colon\ k = 1, \ldots, t\}$. But for each k, $1 \le k \le t$,

$$x_j f_k = x_j e_1 f_k = x_j e_k \in P_j.$$

Thus $Q_j \subset P_j$ for $j = 1, \ldots, r$. Then, since $V = \bigcup_{j=1}^{r} Q_j = \bigcup_{i=1}^{r} P_i$, it is clear that $Q_j = P_j$ for $j = 1, \ldots, r$ and thus, $\pi_1 = \pi_2$. Hence K is a right group and the theorem is proved. ■

Combining this with Theorem 1, we have

Corollary 1. *If the Lie algebra $\mathcal{L}$ of a coloring is solvable, then there exists a coloring semigroup of the graph that contains a synchronizing instruction. In other words, the road coloring conjecture holds for Lie-solvable graphs.*

3. Lie algebra generated by idempotents

In this section $\mathfrak{g} = \mathfrak{g}(x, y)$ will denote the Lie algebra generated by two idempotents x and y. We will find a spanning set for $\mathfrak{g}$. Define $\mathcal{A} = \mathcal{A}(x, y)$ to be the associative algebra with identity generated by x and y.

3.1. Identities

We start with two idempotents x and y. Define $u = x - y$ and $v = 1 - (x + y)$, adjoining the identity as necessary. Then $u^2 = x + y - xy - yx$, $v^2 = 1 - x - y + xy + yx$, $vu = xy - yx = [x, y]$, the commutator, so that we have the basic identities

$$u^2 + v^2 = 1, \quad \text{and} \quad uv + vu = 0.$$

Since u and v anti-commute, $[v, u] = 2vu$, so it is natural to modify the Lie product and define, for any elements in the associative algebra, $a \times b = \frac{1}{2}[a, b]$. Thus, we get $u \times v = uv = -v \times u$.

First, note that u and v generate the same associative algebra (with 1) as do x and y. Now let

$$v_0 = -(x + y) = v - 1. \tag{1}$$

Then u and v_0 generate $\mathfrak{g}(x, y)$. But

$$v_0 \times u = \tfrac{1}{2}[v_0, u] = \tfrac{1}{2}[v, u] = vu = v \times u.$$

Next, observe that v^2 and u^2 commute with u and v, and that, generically, the center of $\mathcal{A}$ is generated by 1 and v^2, as will soon become clear.

3.2. Spanning elements

We define three sequences of elements in the algebra generated by x and y. We will show that, in fact, they are Lie elements that span the Lie algebra generated by x and y. For counting dimensions, we assume the *generic case* where these elements are all linearly independent. Later on, we will assume that they are linearly independent up to a particular finite number according to the structure of the containing semigroup.

The Lie algebra starts with u, v_0, and vu. Since, on u, cross-product with v_0 is the same as multiplication by v, we continue to multiply by v to generate spanning elements. We get two classes:

$$U = \{u_n\colon\ u_n = v^{2n}u, n \geq 0\}, \quad \text{and} \quad W = \{w_n\colon\ w_n = v^{2n-1}u, n \geq 1\}.$$

Thus, these are all Lie elements.

Next, let

$$v_n = u \times w_n = -v^{2n-1}u^2 = v^{2n+1} - v^{2n-1} \tag{2}$$

for $n \geq 1$. And set $V = \{v_n\}_{n \geq 0}$, with $v_0 = v - 1$ as defined previously. We will see that $\mathfrak{g}$ is spanned by $U \cup V \cup W$.

Remark 1. Note that $v_0 = v - 1$, $v_1 = v^3 - v$, $\ldots$, so that an alternative basis for V is given by $\tilde{v}_n = \sum_{k=0}^{n} v_k = v^{2n+1} - 1$, for $n \geq 0$.

First, observe that each of these sets individually is abelian. Next, note that the products $u_n u_m$ are in the center of $\mathcal{A}$, and similarly for v's and w's. Finally, note that any element from one of the sets U, $V \setminus \{v_0\}$, or W anticommutes with every element of either of the other two sets.

Since u anticommutes with odd powers of v and commutes with even powers, we get the following multiplication table.

Proposition 1. *The following rules hold:*

1. *For* $n \geq 0$, $m \geq 1$,

$$\begin{aligned} v_0 \times u_n &= w_{n+1}, \\ v_0 \times w_m &= u_m, \\ u_n \times w_m &= v_{n+m}. \end{aligned}$$

2. *For* $m, n \geq 1$, $j \geq 0$,

$$\begin{aligned} v_n \times u_j &= w_{n+j+1} - w_{n+j}, \\ v_n \times w_m &= u_{n+m} - u_{n+m-1}. \end{aligned}$$

So these elements span a Lie algebra, which is thus $\mathfrak{g}(x, y)$.

3.3. Quaternionic algebras

We see that elements of U are of the form cu, where c is in the center of $\mathcal{A}$. Similarly, elements of V are of the form cv and elements of W are of the form cvu. So, a typical representation of the algebra $\mathcal{A}$ would be formed by mapping u and v to $\hat{u}$ and $\hat{v}$ satisfying $\hat{u}^2 + \hat{v}^2 = \mathrm{I}$ and $\hat{u}\hat{v} = -\hat{v}\hat{u}$. Elements of the center map to scalars so that $\hat{u}^2 = a$, and $\hat{v}^2 = b$ are scalars in the image. Associative algebras generated by anticommuting elements u and v satisfying $u^2 = a$, $v^2 = b$, with a and b scalars are well-known as *generalized quaternion algebras.* (Note that the restriction $a + b = 1$ means that we are mapping to split quaternion algebras.) Since our algebras map to quaternion algebras, we have dubbed them “quaternionic algebras”.

4. Lie algebra of a completely simple semigroup

In this section we take two idempotents from a finite, completely simple semigroup, call them e and f. We assume that they have neither the same partition nor the same range. We know that ef is in the local group with the same partition as e and the same range as f. Let p be the order of ef in that group so that $(ef)^p$ is an idempotent, the local identity for that group. Similarly fe is in its corresponding local group. We have

Proposition 2. *If* $(ef)^p$ *is an idempotent, then* $(fe)^p$ *is an idempotent.*

Proof. Let $(ef)^p = e_1$. Since e_1 is an idempotent with the same range as f, we know that $fe_1 = f$. Thus, $(fe)^p f = f(ef)^p = fe_1 = f$. That is, $(fe)^p$ acts as a left identity for all elements in fS, and in particular on $(fe)^p$ itself. Thus, $(fe)^{2p} = (fe)^p(fe)^p = (fe)^p$ is an idempotent. ■

To see how the spanning set looks, we find an expression for u_n in terms of e and f.

Proposition 3. *For $n \geq 0$, $u_n = (ef)^n e - (fe)^n f$.*

Proof. The proof is by induction. For $n = 0$, we have $u = e - f$, as defined. Now multiply both sides of $u_n = (ef)^n e - (fe)^n f$ by $v^2 = 1 - e - f + ef + fe$. On the right we get

$$\begin{aligned}(ef)^n e - (fe)^n f - (ef)^n e + (fe)^{n+1} - (ef)^{n+1} + (fe)^n f \\ + (ef)^{n+1} - (fe)^{n+1} f + (ef)^{n+1} e - (fe)^{n+1}\end{aligned}$$

which simplifies appropriately. ■

Now we see that

$$u_p = v^{2p} u = (ef)^p e - (fe)^p f.$$

Use the fact that $(ef)^p$ is the idempotent in its local group and it has the same partition as does e, i.e., it's in the right-zero semigroup corresponding to that partition. So $(ef)^p e = e$. Similarly, $(fe)^p f = f$. Thus $u_p = u$, i.e.,

$$v^{2p} u = u. \tag{3}$$

So the sequence $\{u_n\}_{n\geq 0}$ is periodic with period p. And it follows that the sequences $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ are periodic with period p as well. Including v_0, we thus have generically $3p + 1$ basis elements.

We see that

Corollary 2. *The element v satisfies $(v^{2p} - 1)^2 = 0$.*

Proof. From $v^{2p}u = u$ follows $v^{2p}u^2 = u^2$ or $(v^{2p} - 1)u^2 = 0$. That is, $(v^{2p} - 1)(v^2 - 1) = 0$. Since $v^{2p} - 1 = (v^2 - 1)(1 + v^2 + \cdots + v^{2p-2})$, multiplying both sides by this last factor yields the result. ■

Corollary 3. *1. The element u satisfies the equation $((1 - u^2)^p - 1)u = 0$. In particular, possible eigenvalues are either 0 or of the form $(1 - \zeta_p)^{1/2}$, with ζ_p a pth root of unity.*

2. The element v has possible eigenvalues $(2p)$th roots of unity.

Observe, then, that v is invertible. In fact, we have

Proposition 4. *Denote idempotents $e' = (ef)^p$ and $f' = (fe)^p$. Then $v(e, f) = 1 - e - f$ and $v(e', f') = 1 - e' - f'$ satisfy $v(e, f)v(e', f') = 1$, i.e.,*

$$v(e, f)^{-1} = v(e', f').$$

Proof. Arranging rows according to elements with the same partition and columns according to ranges we have the 2×2 array

$$\begin{array}{c|c|c|} & R_1 & R_2 \\ \hline \pi_1 & f' & f \\ \hline \pi_2 & e & e' \\ \hline \end{array} \tag{4}$$

Recall that the columns form left-zero semigroups and the rows, right-zero semigroups. Now, multiplying out $(1-e-f)(1-e'-f')$ yields

$$1 - e' - f' - e + ee' + ef' - f + fe' + ff'$$

which simplifies down to 1 using the zero-properties just noted. ■

4.1. Levi-Malcev decomposition and oscillator subalgebra

The *Levi-Malcev* decomposition of a Lie algebra $\mathfrak{g}$ is the decomposition $\mathfrak{g} = \mathcal{G} \oplus \mathcal{I}$ as a direct sum of the solvable radical $\mathcal{I}$ of $\mathfrak{g}$ with a semisimple algebra $\mathcal{G}$, the *Levi factor* ([6, pp. 91–92]). This is a direct sum of linear subspaces where $\mathcal{G}$ acts on $\mathcal{I}$, thereby forming a semidirect product of Lie algebras.

Define the special element

$$\tau = \frac{1}{p} \sum_{n=0}^{p-1} v^{2n}$$

in the center of $\mathcal{A}$.

By periodicity, we have $\tau v^2 u = v^2 \tau u = \tau u$. Iterating and averaging yields

$$\tau^2 u = \tau u$$

so that τ is a projection on $U \cup V \setminus \{v_0\} \cup W$. Note that

$$\tau u^3 = \tau(1 - v^2)u = \tau u - v^2 \tau u = 0.$$

We will use τ to get the Levi-Malcev decomposition of $\mathfrak{g}$.

4.1.1. Solvable radical

By periodicity, $\tau u = \tau u_n$, $n \geq 0$, $\tau v_1 = \tau v_n$, $n \geq 1$, and $\tau w_1 = \tau w_n$, $n \geq 1$. So define

$$\nu = \tau u, \quad \mu = \tau v_1, \quad \omega = \tau w_1,$$

Lie elements in the range of τ. Averaging, we can write

$$\nu = \frac{1}{p} \sum_{n=0}^{p-1} u_n, \quad \mu = \frac{1}{p} \sum_{n=1}^{p} v_n, \quad \omega = \frac{1}{p} \sum_{n=1}^{p} w_n. \tag{5}$$

Note that we are missing v_0. Let

$$\rho = \tau v - 1.$$

We must check that $\rho \in \mathfrak{g}$. In fact,

Proposition 5. *We have*

$$\rho = \frac{1}{p} \sum_{n=0}^{p-1} (p-n) v_n$$

an element of $\mathfrak{g}$.

Proof. Recalling equation (2), it follows that

$$\sum_{j=0}^{n} v_j = v_0 + v^{2n+1} - v = v^{2n+1} - 1.$$

Therefore

$$\frac{1}{p} \sum_{n=0}^{p-1} \sum_{j=0}^{n} v_j = \frac{1}{p} \left(v + v^3 + \cdots + v^{2p-1} \right) - 1 = \tau v - 1.$$

Now, writing this alternatively as

$$p\,\rho = v_0 + (v_0 + v_1) + \cdots + (v_0 + v_1 + \cdots + v_{p-1}) \tag{6}$$

and combining terms gives the desired result. ■

Remark 2. Note that in terms of the alternative basis for V, Remark 1, we have, by equation (6),

$$\rho = \frac{1}{p} \left(\tilde{v}_0 + \tilde{v}_1 + \cdots + \tilde{v}_{p-1} \right) = \frac{1}{p} \sum_{n=0}^{p-1} \tilde{v}_n.$$

Let $\mathcal{I}$ denote the Lie algebra spanned by $\{\nu, \omega, \mu, \rho\}$. We will see that $\{\nu, \omega, \mu\}$ span a Heisenberg subalgebra normalized by ρ, in other words $\mathcal{I}$ is, generically, an *oscillator algebra*, $\mathfrak{osc}$.

Since τ is in the center, $(\tau a) \times (\tau b) = \tau^2 (a \times b)$. So, calculate

$$\nu \times \omega = \tau^2 u \times w_1 = \tau u \times w_1 = \tau v_1 = \mu$$

and

$$\mu \times \nu = \tau^2 v_1 \times u = -\tau v u^3 = 0.$$

Similarly, $\omega \times \mu = 0$. Thus, these elements span a Heisenberg algebra, call it $\mathcal{H}$.

Next, we have $\rho \times \mu = \tau^2 v \times v_1 = 0$, since the v's commute. And

$$\rho \times \nu = \tau^2 v \times u = \tau v u = \tau w_1 = \omega.$$

Similarly, $\rho \times \omega = \nu$. Thus, ρ normalizes $\mathcal{H}$, and commutes with its center, so all four elements together span an oscillator algebra.

So $\mathcal{I}$ is solvable.

Proposition 6. *$\mathcal{I}$ is a solvable ideal.*

Proof. First, note that $\mathcal{H}$ is in the range of τ, while ρ differs by an element of the center from the range of τ. So multiplying by b we consider something of the form $(\tau a) \times b$, where a is either v or an element of $U \cup V \cup W$ and b is either v or u.

First, consider $a \in U \cup V \setminus \{v_0\} \cup W$. Then a has a factor of u so $\tau a = \tau^2 a$ and $\tau a \in \mathcal{I}$. Thus,

$$\tau a \times b = \tau a \times \tau b$$

with $b = u$ or $b = v$. In either case this equals the product of elements from $\mathcal{I}$.

Now let $a = v$. Then $b = v$ commutes with τv while

$$\tau v \times u = \tau v \times \tau u = \rho \times \nu = \omega. \qquad \blacksquare$$

That $\mathcal{I}$ is maximal will be clear from Section 4.1.3.

4.1.2. Identities for v and μ

It is possible that $\mathcal{I}$ has dimension smaller than 4 according to whether μ is zero or not. E.g., in some cases $\mathcal{I}$ reduces to the two-dimensional affine algebra, $\mathfrak{aff}(2)$. We show examples below (Section 5) where $\nu = -\omega$, so $\mu = \nu \times \omega = 0$. And $\rho \times \nu = -\nu$, thus yielding $\mathfrak{aff}(2)$. In this section, we find some identities for v and refine our view of μ.

We start with a general result. First some preparatory calculations.

Lemma 2. *The following relations hold for $n \geq 0$:*

$$\begin{aligned} e\,v^{2n} &= (ef)^n e \qquad e\,v^{2n-1} = -(ef)^n, \\ f\,v^{2n} &= (fe)^n f \qquad f\,v^{2n-1} = -(fe)^n. \end{aligned}$$

Proof. Start with $ev = e(1-e-f) = -ef$, $fv = f(1-e-f) = -fe$, and check $ev^2 = -efv = efe$, $fv^2 = -fev = fef$. Continue inductively, using these relations to reduce one factor of v at a time. For example,

$$ev^{2n-1} = evv^{2n-2} = -efv^{2n-2} = -e(fe)^{n-1}f = -(ef)^n,$$

and similarly for the other cases. $\blacksquare$

Theorem 3. *We have* $\mu = (v^{2p} - 1)/p$.

Proof. First, observe, recalling equations (5) and (2),

$$p\,\mu = \sum_{n=1}^{p} v_n = \sum_{n=1}^{p} (v^{2n+1} - v^{2n-1}) = v^{2p+1} - v. \tag{7}$$

From the above lemma, we have, using the periodicity relations $(ef)^{p+1} = ef$ and $(fe)^{p+1} = fe$,

$$\begin{aligned} pe\mu &= e(v^{2p+1} - v) = -(ef)^{p+1} + ef = 0, \\ pf\mu &= f(v^{2p+1} - v) = -(fe)^{p+1} + fe = 0, \end{aligned}$$

i.e., $e\mu = f\mu = 0$. So $v = 1 - e - f$ yields $v\mu = \mu$. But $\mu = -\tau v u^2$ implies, using $\tau u^3 = 0$,

$$\mu = -\tau v u^2 = v\mu = -\tau v^2 u^2 = -\tau(1-u^2)u^2 = -\tau u^2 = \tau(v^2-1) = (1/p)(v^{2p}-1)$$

from the definition of τ as a geometric sum. ∎

Now Corollary 2 may be recast as $\mu^2 = 0$ and, recalling equation (1), we have the refinement

Corollary 4. *The relation* $(v^{2p} - 1)v_0 = 0$ *holds.*

Proof. In the proof of the theorem, we saw that $v\mu = \mu$. That is, $\mu v_0 = 0$. With $p\mu = v^{2p} - 1$, the result follows. ∎

We note also from the proof that $\mu = -\tau u^2$. The corollary and equation (3) show that μ acts as zero on the entire Lie algebra. Put another way, v^{2p} acts as an identity for multiplication (in $\mathcal{A}$) on elements of the Lie algebra.

If ν and ω are linearly dependent, then $\nu \times \omega = \mu$ is automatically zero, which thus entails $v^{2p} = 1$. Finally, $\mu = 0$ implies that ν and ω commute, so that the subalgebra $\mathcal{H}$ becomes abelian.

Remark 3. 1. Note that v intertwines e and f, i.e., $ve = fv$ and $ev = vf$. And similarly for v^{-1}.

2. Referring to Proposition 4, we have $e' = (ef)^p$ as the idempotent in its group. And $f' = (fe)^p$. Setting $n = p$ in the equations of Lemma 2, we have $ev^{2p} = (ef)^p e = e'e = e$, $fv^{2p} = (fe)^p f = f'f = f$ and the two right side equations yield

$$e' = -ev^{-1}, \qquad f' = -fv^{-1}.$$

So $e'f' = ev^{-1}fv^{-1} = ev^{-2}$ and hence $(e'f')^p = ev^{-2p} = e$, and similarly for $f'e'$. In other words, $e'f'$, $f'e'$, ef, and fe all have the same exponent.

3. We can now verify that e' and f' are in $\mathfrak{g}(e,f)$, the Lie algebra generated by e and f. Start with $-2f = v_0 + u = v + u - 1 = v + u - u^2 - v^2$. Now use periodicity to get

$$\begin{aligned} -2f &= v + v^{2p}u - v^{2p}u^2 - v^2 = v - v^{2p-1}uv - v^{2p-1}u^2v - v^2 \\ &= (1 - w_p + v_p - v)v = (-w_p + v_p - v_0)v \end{aligned}$$

so that $2f' = -2fv^{-1} = -w_p + v_p - v_0 \in \mathfrak{g}(e,f)$. Similarly for e', start with $-2e = v_0 - u$ to get $2e' = w_p + v_p - v_0$.

4.1.3. Root-space decomposition

We will diagonalize the linear map $v_0\times$ acting on $\mathfrak{g}$. It maps $V \to 0$, $U \to W$ and $W \to U$. On U and W, this map is the same as left multiplication by v. We calculate the matrix, $\check{v}$, of this map on the invariant subspace spanned by $U \cup W$.

Remark 4. On $\mathcal{I}$, $v_0\times$ is the same as $\rho\times$. That is, $v_0 \times \rho = v_0 \times \mu = 0$ while ω and ν are switched.

Note that Corollary 4 shows that the eigenvalues of $\check{v}$ are $(2p)$th roots of unity. On the span of $U \cup W$, order the bases U first, then W. Using Proposition 1, we find the block form

$$\check{v} = \begin{bmatrix} 0 & X \\ I & 0 \end{bmatrix}$$

where I is the $p \times p$ identity and X is the $p \times p$ circulant

$$X = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The corresponding block form for a vector is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \sum_{i=0}^{p-1} (a_{i+1}u_i + b_{i+1}w_{i+1}) \tag{8}$$

with indices on a and b running from 1 to p.

Let λ run through the distinct pth roots of unity, and denote corresponding $(2p)$th roots of unity by $\pm\sqrt{\lambda}$. Now check that eigenvectors of $\check{v}$ are of the block form, as in equation (8),

$$E_\lambda^\pm = \begin{bmatrix} \pm\sqrt{\lambda}\,\xi_\lambda \\ \xi_\lambda \end{bmatrix} \tag{9}$$

where $X\xi_\lambda = \lambda\xi_\lambda$, i.e., ξ_λ is an eigenvector of X with eigenvalue λ, which we choose as

$$\xi_\lambda = \begin{pmatrix} 1 \\ \bar\lambda \\ \vdots \\ \bar\lambda^{p-1} \end{pmatrix}$$

with the bar indicating complex conjugate.

So, in terms of u's and w's, scaling out a factor of p, we have, from equations (8) and (9)

$$\begin{aligned} E_\lambda^+ &= \frac{1}{p}\sum_{i=0}^{p-1} \bar\lambda^i(\sqrt{\lambda}\,u_i + w_{i+1}) \\ &= \frac{1}{p}\sum_{i=0}^{p-1} \bar\lambda^i(\sqrt{\lambda}\,v^{2i} + v^{2i+1})u \\ &= (v+\sqrt{\lambda})\,\frac{1}{p}\sum_{i=0}^{p-1} \bar\lambda^i\, v^{2i}u \\ &= \frac{1}{p}\,(v+\sqrt{\lambda})\frac{1-v^{2p}}{1-\bar\lambda v^2}\,u, \end{aligned}$$

the quotient serving as an abbreviation for the corresponding polynomial in v. Note that if $\lambda = 1$, then we get $E_1^+ = (v+1)\tau u = \omega + \nu$.

We thus have

Theorem 4. *The adjoint action of v_0 on $\mathfrak{g}$ has the following root-space decomposition, with λ running through pth roots of unity.*

1. *For $\lambda = 1$,*

$$E_1^\pm = (v\pm 1)\tau u = \begin{cases} \omega+\nu, & \text{for } +1, \\ \omega-\nu, & \text{for } -1. \end{cases}$$

2. *For $\lambda \neq 1$,*

$$E_\lambda^\pm = \frac{(v\pm\sqrt{\lambda})}{p}\frac{1-v^{2p}}{1-\bar\lambda v^2}\,u.$$

4.1.4. Cartan subalgebra and Levi factor

For $\lambda = 1$, we have ρ as Cartan element, with $\rho \times E_1^\pm = \pm E_1^\pm$. And

$$E_1^+ \times E_1^- = 2\mu,$$

these elements thus recovering $\mathcal{I}$.

Generally, we have, with σ denoting either $+$ or $-$,

$$\begin{aligned} p\,E^{\pm}_{\lambda_1}E^{\sigma}_{\lambda_2} &= (v \pm \sqrt{\lambda_1})\frac{1-v^{2p}}{1-\bar{\lambda}_1 v^2}\,u\,E^{\sigma}_{\lambda_2} \\ &= u\,(-\sigma\sqrt{\lambda_2} \pm \sqrt{\lambda_1})\frac{1-{\lambda_2}^p}{1-\bar{\lambda}_1\lambda_2}\,E^{\sigma}_{\lambda_2} \qquad (10) \end{aligned}$$

which thus equals zero unless $\bar{\lambda}_1\lambda_2 = 1$. Thus the root spaces along with their corresponding Cartan elements form a direct product.

Now, for the Cartan elements associated with $\lambda \neq 1$, we have

$$\begin{aligned} E^{+}_{\lambda}E^{-}_{\lambda} &= u\,\frac{(-v+\sqrt{\lambda})}{p}\,\frac{1-v^{2p}}{1-\bar{\lambda}v^2}\,E^{-}_{\lambda} \\ &= 2u\,\sqrt{\lambda}\,E^{-}_{\lambda} \qquad (11) \end{aligned}$$

via the evaluation $\left.\frac{1-v^{2p}}{1-\bar{\lambda}v^2}\right|_{v^2=\lambda} = p$. Similarly,

$$E^{-}_{\lambda}E^{+}_{\lambda} = -2u\,\sqrt{\lambda}\,E^{+}_{\lambda},$$

thus, we have the Cartan element

$$\begin{aligned} H_{\lambda} &= [E^{+}_{\lambda}, E^{-}_{\lambda}] = 2u\,\sqrt{\lambda}(E^{+}_{\lambda} + E^{-}_{\lambda}) \\ &= \frac{4\sqrt{\lambda}}{p}\,uv\,\frac{1-v^{2p}}{1-\bar{\lambda}v^2}\,u \\ &= \frac{-4\sqrt{\lambda}}{p}\,\frac{1-v^{2p}}{1-\bar{\lambda}v^2}\,v(1-v^2). \end{aligned}$$

And equation (10) checks that $H_{\lambda_1}E^{\pm}_{\lambda_2} = 0$ if $\lambda_1 \neq \lambda_2$. Hence, evaluating as in equation (11),

$$\begin{aligned} H_{\lambda} \times E^{+}_{\lambda} &= -4\sqrt{\lambda}\,\sqrt{\lambda}(1-\lambda)\,E^{+}_{\lambda} = -4\lambda(1-\lambda)\,E^{+}_{\lambda}, \\ H_{\lambda} \times E^{-}_{\lambda} &= 4\lambda(1-\lambda)\,E^{-}_{\lambda} \end{aligned}$$

which is not zero if $\lambda \neq 1$. That is, each root-space for $\lambda \neq 1$ generates a copy of $\mathfrak{sl}(2)$. The Levi-Malcev decomposition is thus completely described: the semisimple part $\mathcal{G}$ is homomorphic to a direct sum of $p-1$ copies of $\mathfrak{sl}(2)$ the solvable radical $\mathcal{I}$ corresponds to $\lambda = 1$, with ρ as the associated Cartan element.

4.2. Lie-solvable kernels and right groups

Now consider $\mathcal{L} = \mathfrak{g}(\mathcal{C})$ for some coloring $\mathcal{C}$. If the kernel is not a right group, then we can find e and f such that ef is not itself an idempotent, so that $p > 1$. Thus $\mathcal{G}$ is nontrivial and $\mathcal{L}$ is not solvable. In other words, $\mathcal{L}$ solvable implies that the kernel is a right group. So we have another proof of Theorem 2. But note that we are using the fact that a direct product must be a right group.

For right groups we have the following structure. In this case e and f have the same partition, with $ef = f$ and $fe = e$. Then $u^2 = 0$ and $v^2 = 1$. In this case, ef and fe are themselves idempotents, $p = 1$, and $\mathcal{G}$ is empty. We note here the general identity, whether or not we are in a right group,

Proposition 7. *For any idempotents e and f, with $v = 1 - (e + f)$, we have*

$$(ef)^2 = v^2\, ef.$$

From this it is immediate that if $v^2 = 1$, then ef is an idempotent. And noting that v is symmetric in e and f, then so is fe. Observe that if $v^2 = 1$, then τ is the identity, so has zero nullspace. And $\mu = \tau v_1 = v_1 = -vu^2 = 0$, $\nu = u = -vu = -\omega$. In fact,

$$u \times v = fe - ef = e - f = u$$

so that $\mathfrak{g}(e, f)$ is $\mathfrak{aff}(2)$ (cf. 4.1.2).

We can say a bit more. Let $\{e_i\}_{1\le i\le t}$ be the set of idempotents of a right group, thus forming a right-zero semigroup. Now, for any k, $i \neq j$,

$$[e_k, e_i - e_j] = e_i - e_j$$

independent of k. Thus setting $e_{ij} = e_i - e_j$, we have, for all pairs i, j and m, n

$$[e_{ij}, e_{mn}] = 0.$$

Since $[e_k, e_l] = e_{lk}$, $\mathfrak{g}(e_1, e_2, \dots, e_t)$, the Lie algebra generated by the idempotents, has $\mathfrak{g}'$ abelian. So $\mathfrak{g}$ is a two-step solvable Lie algebra.

5. Examples

First we introduce an interesting class of examples where a nontrivial sandwich function is guaranteed for at least one coloring. Then we look at certain recolorings that illustrate various possibilities for the corresponding kernels' Lie algebraic properties.

5.1. An infinite class of strongly connected, aperiodic digraphs

The following construction produces an infinite class of examples, which will be shown to be strongly connected and aperiodic. In addition, this class will have at least one coloring which generates a semigroup whose kernel has a non-trivial sandwich function. Let $V = \{1, 2, \ldots, 2k\}$ for any integer $k \geq 2$. We will define a completely simple semigroup of transformations by choosing two range sets R_1, R_2 and two partitions π_1, π_2 of V that are each compatible with R_1, R_2. Let $R_1 = \{1, 3, \ldots, 2k-1\}$ and $R_2 = \{2, 4, \ldots, 2k\}$. Set the first partition $\pi_1 = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}\}$. Clearly, π_1 is compatible with both R_1 and R_2. The first generator will be chosen to have partition π_1 and range R_1. To this end, let

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2k-3 & 2k-2 & 2k-1 & 2k \\ 3 & 3 & 5 & 5 & \cdots & 2k-1 & 2k-1 & 1 & 1 \end{pmatrix}$$

i.e., r maps 1 to 3, 2 to 3, etc.

Now let $(i_1, i_2, \ldots, i_{k-1})$ be a permutation of the even integers $\{4, \ldots, 2k\}$ and define the second generator by

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & 2k-2 & 2k-1 & 2k \\ 2 & 4 & i_1 & 6 & i_2 & \cdots & 2k & i_{k-1} & 2 \end{pmatrix}$$

which determines π_2 as the corresponding induced partition.

Remark 5. It is often convenient to use "transformation notation," writing the above functions, e.g., as $r = [3, 3, 5, 5, \ldots, 2k-1, 2k-1, 1, 1]$ and $b = [2, 4, i_1, 6, i_2, \ldots, 2k, i_{k-1}, 2]$.

Denote the corresponding $\{0, 1\}$ matrices $[r]$ and $[b]$ respectively. If we let $A = [r] + [b]$, then A is the adjacency matrix of a strongly connected and aperiodic graph. Strong connectivity is clear from the definition of r and b, since r provides a path to each of the odd integers and b to each of the even integers. To show aperiodicity, notice that r^k is an idempotent that maps 1 to itself, thus 1 has a cycle of length k. In addition, br^k also maps 1 to 1, thus $\mathrm{per}(1) = \gcd(k, k+1) = 1$.

Now $S = \langle r, b\rangle$ is a completely simple semigroup. Showing that the sandwich function is nontrivial is equivalent to showing that there exists idempotents e, f such that $(ef)^2 \neq ef$. Recall that a transformation is idempotent if and only if it is the identity on its range. Let $e = r^k$ and $f = b^k$. Then $2ef = 1f = 2k \neq 2$, as long as $k \geq 2$. Thus ef is not an idempotent, and S has a nontrivial sandwich function.

It is worth noting that this class of digraphs has uniform indegree as well as uniform outdegree. Kari [7] has shown that the road coloring conjecture is true for this class of digraphs.

5.2. Examples

There are two cases of the above construction for $k = 3$. We will look at the associated kernel and some recolorings.

Notation. To denote the structure of a Lie algebra $\mathfrak{g}$ with Levi decomposition $\mathcal{G} \oplus \mathcal{I}$, we use the notation $\mathfrak{g} \sim \mathbf{d} \oplus \mathbf{n}/n_1$ where $d = \dim \mathcal{G}$, $n = \dim \mathcal{I}$ and $n_1 = \dim \mathcal{I}'$. We indicate by $\bullet/n$ a solvable algebra $\mathfrak{g}$ of dimension $n+1$, with $\mathfrak{g}'$ n-dimensional abelian. For example, $\bullet/1 \approx \mathfrak{aff}(2)$.

Remark 6. The Lie algebra calculations were carried out using GAP.

Example 1. a. Let $r = [3,3,5,5,1,1]$, $b = [2,4,4,6,6,2]$. The kernel has the shape, cf. the diagram (4),

$$\begin{array}{rcc} & 135 & 246 \\ 12|34|56 & e_1 & e_2 \\ 16|23|45 & e_3 & e_4 \end{array}$$

with the e_i's idempotents for the corresponding groups, which in this case are isomorphic to C_3, the cyclic group of order 3. Recall that in each cell, the idempotent is determined as the transformation acting as the identity on its range. Here $p = 3$, $\mathcal{L} = \mathcal{K} \sim \mathbf{6} \oplus \mathbf{2}/1$, corresponding to the Levi decomposition $\mathcal{G} \oplus \mathcal{I}$, with $\mathcal{G}$ isomorphic to two copies of $\mathfrak{sl}(2)$ and $\mathcal{I} \approx \mathfrak{aff}(2)$, i.e., $\mu = 0$ in this case.

b. Recolor to $r = [3,3,5,6,6,2]$, $b = [2,4,4,5,1,1]$. Then $\mathcal{L} \sim \mathbf{8} \oplus \mathbf{4}/2$, and $\mathcal{K} \sim \mathbf{8} \oplus \bullet/2$. The kernel has the shape

$$\begin{array}{rccc} & 14 & 25 & 36 \\ 123|456 & e_1 & e_2 & e_3 \\ 126|345 & e_4 & e_5 & e_6 \\ 156|234 & e_7 & e_8 & e_9 \end{array}$$

The local groups are C_2's. Denote, e.g., $[1 \times 5]$ the block of four cells with diagonal containing e_1 and e_5, with $\mathfrak{g}(1 \times 5)$ the corresponding Lie algebra. Then $[1 \times 5]$ is a direct product, $p = 1$, where the product of idempotents is an idempotent, e.g., $e_1 e_5 = e_2$, with $\mathfrak{g}(1 \times 5)$ an oscillator algebra. While $[1 \times 6]$ has $p = 2$, with $\mathfrak{g}(1 \times 6) \sim \mathbf{3} \oplus \bullet/1 \approx \mathfrak{sl}(2) \oplus \mathfrak{aff}(2)$.

Example 2. a. Let $r = [3,3,5,5,1,1]$, $b = [2,4,6,6,4,2]$. The kernel has the shape

$$\begin{array}{rcc} & 135 & 246 \\ 12|34|56 & e_1 & e_2 \\ 16|25|34 & e_3 & e_4 \end{array}$$

The local groups are now S_3 — symmetric groups. Here $p = 2$, $\mathcal{L} = \mathcal{K} \sim \mathbf{8} \oplus \bullet/4$, with $\mathbf{8} \approx \mathfrak{sl}(3)$. The Lie algebra

$$\mathfrak{g}(1 \times 4) \sim \mathbf{3} \oplus \bullet/1 \approx \mathfrak{sl}(2) \oplus \mathfrak{aff}(2)$$

b. Recoloring, $r = [3,4,5,5,1,1]$, $b = [2,3,6,6,4,2]$.

	135	145	236	246
12|34|56	e_1	e_2	e_3	e_4
16|25|34	e_5	e_6	e_7	e_8

Here $\mathcal{L} \sim \mathbf{8} \oplus \mathbf{10}/8$. Local groups are S_3. For $\mathfrak{g}$, the Lie algebra generated by the idempotents, we have $\mathfrak{g} \sim \mathbf{3} \oplus \bullet/2$. The Lie algebra $\mathfrak{g}(3 \times 8) \sim 0 \oplus \bullet/2$, which thus has $\mu = 0$, but is not $\mathfrak{aff}(2)$.

c. $r = [3,4,5,6,1,2]$, $b = [2,3,6,5,4,1]$. These are two permutations generating a group of order 36. The Lie algebra they generate is $\mathfrak{so}(5)$ plus 2-dimensional abelian.

d. $r = [3,4,6,5,1,2]$, $b = [2,3,5,6,4,1]$. These permutations generate the full symmetric group S_6. The Lie algebra they generate is isomorphic to $\mathfrak{gl}(5)$.

e. $r = [3,4,6,5,4,1]$, $b = [2,3,5,6,1,2]$. The kernel is a right group, but $\mathcal{L} \sim \mathbf{8} \oplus \mathbf{13}/11$ is not solvable, the $\mathbf{8} \approx \mathfrak{sl}(3)$. The kernel has the shape, only one partition,

	12	14	15	32	34	35	62	64	65
136|245	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9

The Lie algebra generated by the idempotents $\mathfrak{g} \sim \bullet/4$, with $\mathcal{K} \sim 0 \oplus \mathbf{10}/8$.

6. Conclusion

Future work will focus on a number of areas. While solvability is a condition that is immediately verifiable for a given edge-coloring, it would be useful to determine an equivalent graph-theoretic condition. Further study of the Lie algebras of the coloring transformations and of the kernels of the associated coloring semigroups is clearly warranted. The group generated by the v-operators of 2×2 arrays within the kernel (cf. diagram (4), but with cells not necessarily contiguous) and its relationship to both the group $G = e_0 K e_0$ of the Rees product and the subgroup within G generated by the sandwich function are areas of particular interest.

Acknowledgments

The first calculations, using GAP, of Lie algebras generated by coloring transformations were developed by J. Gill as part of his Master's Thesis [4]. He observed that in every case where the Lie algebra was solvable the kernel was a right group.

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