Date: May 21, 2005
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Subject: JMAA-04-1010R1: Final Decision
Ms. No.: JMAA-04-1010R1
Title: Convolution Products of Probability Measures on Completely Simple Semigroups
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Dear Professor Budzban,

We are pleased to inform you that your manuscript referenced above has been accepted for publication in the Journal of Mathematical Analysis and Applications.

Many thanks for submitting your fine paper to the Journal of Mathematical Analysis and Applications. We look forward to receiving additional papers from you in the future.

With kind regards,

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CONVOLUTION PRODUCTS OF PROBABILITY MEASURES ON COMPLETELY SIMPLE SEMIGROUPS

GREGORY BUDZBAN

Abstract. Convolution products of probability measures are considered in the context of completely simple semigroups. Given a sequence of measures $(\mu_n) \subset \text{Prob}(S)$, where S is a finite completely simple semigroup, results are proven which (1) relate the supports of the measures in the sequence to the supports of the tail limit measures, and (2) determine necessary and sufficient conditions for convergence of the convolution products in the case of rectangular groups. An example showing how the theory can be used to analyze the convergence behavior of non-homogeneous Markov chains is included.

1. Introduction

This paper continues an ongoing program of analyzing the convergence of convolution products of probability measures on groups and semigroups. The convergence theory for powers, μ^n, of a single probability measure is largely complete, even in structures as general as locally compact semigroups, (for details, see [5]). The situation changes completely for products of non-identical measures. The specifics concerning recent progress will be given below.

Let (μ_n) be a sequence of probability measures on a topological semigroup, S. For a Borel set, $B \subset S$, define

$$\mu_1 * \mu_2(B) = \int \mu_1(Bx^{-1})\mu_2(dx) \tag{1}$$

where $Bx^{-1} = \{y \in S \,|\, yx \in B\}$. We seek to determine under what conditions the convolution product

$$\mu_{k,n} = \mu_{k+1} * \mu_{k+2} * \cdots * \mu_n$$

will converge weakly, as $n \to \infty$, for all $k \geq 0$.

Center and Mukherjea initiated the study of convolution products in semigroups in [4]. Mukherjea continued the analysis in [7] where he analyzed their behavior in compact Abelian semigroups, finding certain necessary conditions for weak convergence. The program was expanded to compact semigroups in [2] where, among other things, verifiable sufficient conditions for convergence were found.

Ruzsa continued the work in [8] where he found an extremely useful sufficient condition for discrete Abelian semigroups, which was extended to certain discrete non-Abelian semigroups in [3].

Complete verifiable necessary and sufficient conditions in this area are rare. This was accomplished for finite Abelian semigroups in [1]. The main result of this current paper completes the problem for a class of non-Abelian semigroups known as rectangular groups.

The organization of this paper will be as follows. After this introduction, there will be a section that provides background information concerning notation and some semigroup theory. The section that includes the main results will follow and the paper will conclude with an application to non-homogeneous Markov chains and a discussion of possible future directions for the program.

2. Some Prerequisites

All of the definitions and results of this section can be stated in greater generality but since the main results of this paper are for finite semigroups, we will work in this context.

Suppose S is a finite simple semigroup. Let $E(S)$ be the set of idempotents in S. A partial order on $E(S)$ can be defined by saying that for $e_1, e_2 \in E(S)$, $e_1 \leq e_2$ if and only if $e_1e_2 = e_2e_1 = e_1$. Choose an idempotent, e, minimal with respect to this order. Let $X = E(Se)$, $G = eSe$, $Y = E(eS)$. It can be shown that the mapping $\phi_e : S \to X \times G \times Y$ defined by

$$\phi_e(s) = \big(s(ese)^{-1},\ ese,\ (ese)^{-1}s\big)$$

is an isomorphism when $X \times G \times Y$ is given the product

$$(x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1, g_1(y_1x_2)g_2, y_2).$$

Notice that $y_1x_2 \in (eS)(Se) = eSe = G$. This mapping from $Y \times X$ into G, $(y, x) \mapsto yx$, plays an important role in the structure theory of these semigroups and is called the "sandwich function".

The product structure $X \times G \times Y$ is called a Rees product and one of the earliest results in the field of semigroup theory was by Suskevitch (later generalized by Rees) showing that for finite semigroups, the existence of this isomorphic product structure was equivalent to being simple. Since this theory was extended to an infinite setting where the existence of a minimal idempotent is not guaranteed, those semigroups that are simple and have a minimal idempotent are called completely simple.

An important special class of completely simple semigroups are those where the set of idempotents form a subsemigroup. In this case, the sandwich function is said to be trivial since $yx = e = e^2 \in G$ for all $(y, x) \in Y \times X$. This class of completely simple semigroups are called "rectangular groups".

In what follows, if no confusion will result, we will write for $s \in S$, $s = (s_1, s_2, s_3)$ when $\phi_e(s) = (s_1, s_2, s_3)$. In addition, when $\phi_e(S) = X \times G \times Y$ we will write $S \cong X \times G \times Y$. Note, in this finite setting,

$$\mu_1 * \mu_2(B) = \sum_{x \in S} \mu_1(Bx^{-1})\mu_2(x).$$

3. Main Results

We begin our analysis of the problem by finding conditions necessary for convergence, which also characterize the relationship between the supports of the limit points of the sequence (μ_n) and supports of the "tail limits" of (μ_n). A measure λ is called a tail limit of (μ_n) if λ is a limit point of (ν_k), where for each k, $\mu_{k,n_i} \to \nu_k$ as $n_i \to \infty$. In a paper fundamental to this area, Csiszar [5] showed that in compact semigroups, given any sequence (n_i), there is a subsequence $(m_i) \subset (n_i)$ such that $\mu_{k,m_i} \to \nu_k$ and also $\nu_{m_i} \to \nu_\infty = \nu_\infty^2$. Thus, not only tail limits, but tail idempotents are easy to produce.

Theorem 1. *Let* $S \cong X \times G \times Y$ *be a finite, completely simple semigroup. Let* $(\mu_n) \subset P(S)$. *Suppose* $\mu_{k,n} \to \nu_k$ *for all* $k \geq 0$, *as* $n \to \infty$. *Define* $S_1 = \bigcup_\lambda supp(\lambda)$, λ *a limit point of* (ν_k). *Then,*

1. $S_1 = X_1 \times H_1 \times Y_1$ *where* $supp(\lambda) = X_\lambda \times H_1 \times Y_1$
2. $X_1 = \{x \in X | \limsup_n \mu_n(xS) > 0\}$
3. $\lim_n \mu_n(Sy)$ *exists for all* $y \in Y$
4. $Y_1 = \{y \in Y | \lim_n \mu_n(S_y) > 0\}$.

Proof. Statement (1) is well known and stated here only for convenience. For a proof, see [2]

For (2), notice the following:

$$z^{-1}(yS) = \begin{cases} S & z \in yS \\ \emptyset & z \notin yS. \end{cases}$$

Thus

$$\begin{aligned}\mu_{k,n}(xS) &= \sum_{z} \mu_{k+1,n}\big[z^{-1}(xS)\big]\mu_{k+1}(z)\\ &= \sum_{z\in xS} \mu_{k+1,n}\big[z^{-1}(xS)\big]\mu_{k+1}(z)\\ &= \mu_{k+1}(xS).\end{aligned}$$

Thus, since $\mu_{k,n} \to \nu_k$

$$\nu_k(xS) = \mu_{k+1}(xS).$$

Now let $x_1 \in X_1$. Then for some limit point λ of (ν_k), $\lambda(\{x_1\} \times H_1 \times Y_1) > 0$.

If $\nu_{k_i} \to \lambda$, then $\mu_{k_i+1}(xS) = \nu_{k_i}(xS) \to \lambda(\{x_1\} \times H_1 \times Y_1)$ and from this it is clear that

$$\limsup_n \mu_n(x_1S) > 0.$$

Now if $\limsup_n \mu_n(x_1S) > 0$, there exists a subsequence (n_i) such that, eventually with respect to i, for some $\delta > 0$

$$\nu_{n_i}(x_1S) = \mu_{n_{i+1}}(x_1S) > \delta > 0.$$

Then, clearly, there exists a limit point λ of (ν_{n_i}) such that

$$\{x_1\} \times H_1 \times Y_1 \subset \operatorname{supp}(\lambda) \subset X_1 \times H_1 \times Y_1,$$

thus (2) is shown.

For (3), notice that

$$(Sy)z^{-1} = \begin{cases} S & z \in Sy\\ \emptyset & z \notin Sy.\end{cases}$$

Thus,

$$\begin{aligned}\mu_{k,n}(Sy) &= \sum_{z\in Sy} \mu_{k,n-1}\big[(Sy)z^{-1}\big]\mu_n(dz)\\ &= \mu_n(Sy).\end{aligned}$$

Then $\lim_{n\to\infty} \mu_n(Sy) = \lim_{n\to\infty} \mu_{k,n}(Sy) = \nu_k(Sy)$.

As for (4), note that for any k, $\nu_k(Sy) = \lim_{n\to\infty} \mu_n(Sy)$. Thus, if λ is any limit point of (ν_k), $\lambda(Sy) = \lim_{n\to\infty} \mu_n(Sy)$. □

The next result continues investigating the relationship between the limit points of (μ_n) and the union of the support of the tail limits. It finds that the support of any limit point of (μ_n) must be contained in the subsemigroup S_1, constructed from the supports of the tail limits of (μ_n).

Theorem 2. *Let* $S \cong X \times G \times Y$ *be a finite, completely simple semigroup. Let* $(\mu_n) \subset P(S)$*. Suppose* $\mu_{k,n} \to \nu_k$*, for all* $k \geq 0$*, as* $n \to \infty$*. Then* $\lim_n \mu_n(X_1 \times H_1 \times Y_1) = 1$*.*

Proof. We must analyze the structure of the sets $(X_1 \times H_1 \times Y_1)z^{-1}$ in order to understand the behavior of the convolution products.

Let $z = (z_1, z_2, z_3) \in X_1 \times H_1 \times Y_1$. Clearly, $X_1 \times H_1 \times Y_1 \subset (X_1 \times H_1 \times Y_1)z^{-1}$ in this case. Suppose $y \in (X_1 \times H_1 \times Y_1)z^{-1}$. Under what conditions will $y \notin X_1 \times H_1 \times Y_1$?

Let $y = (y_1, y_2, y_3)$. If $y_3 \in Y_1$, then $y_3 z_1 \in H_1$, which implies that $y_2 \in H_1$. Thus if $y_3 \in Y_1$, then $y \in X_1 \times H_1 \times Y_1$. Therefore, suppose $y_3 \notin Y_1$. Then,

$$y_2 \in H_1(y_3 z_1)^{-1} \subset \bigcup_{h \in Y_1^c X_1} H_1 h^{-1}.$$

We have shown, for $z \in X_1 \times H_1 \times Y_1$,

$$(X_1 \times H_1 \times Y_1)z^{-1} \subset X_1 \times H_1 \times Y_1 \cup \bigcup_{h \in Y_1^c X_1} X_1 \times H_1 h^{-1} \times Y_1^c.$$

Since $X_1 \times H_1 \times Y_1 = \bigcup \operatorname{supp}(\lambda)$ where this union is over the limit points, λ, of (ν_k), then for $\varepsilon > 0$

$$\nu_k(X_1 \times H_1 \times Y_1) > 1 - \frac{\varepsilon}{4}$$

for all $k \geq K_1(\varepsilon)$.

Now

$$\mu_{k,n} = \mu_{k+1} * \mu_{k+1,n},$$

thus

$$\nu_k = \mu_{k+1} * \nu_{k+1}.$$

Hence for $k \geq K_1(\varepsilon)$

$$\begin{aligned}
1-\frac{\varepsilon}{4} &< \nu_k(X_1 \times H_1 \times Y_1) \\
&= \sum_z \mu_{k+1}\big[\big(X_1 \times H_1 \times Y_1\big) z^{-1}\big]\ \nu_{k+1}(z) \\
&= \sum_{z\in X_1\times H_1\times Y_1} \mu_{k+1}\big[(X_1 \times H_1 \times Y_1) z^{-1}\big] \nu_{k+1}(z) \\
&\qquad + \sum_{z\in (X_1\times H_1\times Y_1)^c} \mu_{k+1}\big[(X_1 \times H_1 \times Y_1) z^{-1}\big] \nu_{k+1}(z).
\end{aligned}$$

Thus

$$\begin{aligned}
1-\frac{\varepsilon}{2} &< \sum_{X_1\times H_1\times Y_1} \mu_{k+1}\big[\big(X_1 \times H_1 \times Y_1\big) z^{-1}\big] \nu_{k+1}(z) \\
&\leq \mu_{k+1}(X_1 \times H_1 \times Y_1)\nu_{k+1}(X_1 \times H_1 \times Y_1) \\
&\qquad + \mu_{k+1}\big(\bigcup_{h\in Y_1^c X_1} X_1 \times H_1 h^{-1} \times Y_1^c\big)\nu_k(X_1 \times H_1 \times Y_1)
\end{aligned}$$

but by (3) of Theorem 1, $\lim\limits_{k\to\infty} \mu_{k+1}\big(\bigcup\limits_{h\in Y_1^c X_1} X_1 \times H_1 h^{-1} \times Y_1^c\big) = 0.$

Thus for $k \geq K_2(\varepsilon) \geq K_1(\varepsilon)$

$$\mu_{k+1}\big(\bigcup_{h\in Y_1^c X_1} X_1 \times H_1 h^{-1} \times Y_1\big) < \frac{\varepsilon}{2}\,.$$

So that, for $k \geq K_2(\varepsilon)$

$$\begin{aligned}
1-\varepsilon &< \mu_{k+1}(X_1 \times H_1 \times Y_1)\nu_{k+1}(X_1 \times H_1 \times Y_1) \\
&< \mu_{k+1}(X_1 \times H_1 \times Y_1).
\end{aligned}$$

We have thus shown that

$$\lim_n \mu_n(X_1 \times H_1 \times Y_1) = 1.$$

$\square$

Our final result determines necessary and sufficient conditions for the convergence of the convolution products $\mu_{k,n}$ for the case of rectangular groups.

Theorem 3. *Let* $S \cong X \times G \times Y$ *be a finite completely simple semigroup with* $YX = \{e\}$*,* e *the identity of* G*. Suppose* $(\mu_n) \subset P(S)$*. Then* $\mu_{k,n} \to \nu_k$ *weakly for all* $k \geq 0$ *iff*

(1) *There exists a subgroup* $H \subset G$ *such that*

$$\sum 1 - \mu_n(X \times H \times Y) < \infty;$$

(2) *For all sequences* $(h_n) \subset G$ *and subgroups* $H_1 \subset H$

$$\sum 1 - \mu_n(X \times h_n H_1 h_{n+1}^{-1} \times Y) = \infty;$$

(3) *For each* $x \in S$*,* $\lim\limits_{n\to\infty} \mu_n(Sx)$ *exists.*

Proof. Suppose $\mu_{k,n} \to \nu_k$ for $k \geq 0$. Part (1) holds trivially and part (3) follows from Theorem 1 of this paper. To prove (2), assume $\sum 1 - \mu_n(X \times h_n H_1 h_{n+1}^{-1} \times Y) < \infty$ for some sequence of elements $(h_n) \subset G$ and some subgroup H_1 of H, where H is the smallest subgroup of G to satisfy (1).

Notice that

$$(X_1 \times h_1 H_1 h_2^{-1} \times Y)(X \times h_2 H_1 h_3^{-1} \times Y)$$

$$= X_1 \times h_1 H_1 h_3^{-1} \times Y.$$

Since (h_k) has only a finite number of distinct elements, $h_k = h$ for infinitely many k. Choosing a subsequence (n_i) of these n such that for some limit point λ of (ν_k),

$$\mu_{n_i,n_{i+1}} \to \lambda$$

we have that $\lambda(X \times hH_1h^{-1} \times Y) = 1$ due to the fact that

$$\lim_{k\to\infty} \left[\inf_n \mu_{k,n}(X \times h_k H_1 h_n^{-1} \times Y)\right] = 1.$$

But this implies that $H \subset hH_1h^{-1}$, which is a contradiction to the minimality of H.

Now suppose (1), (2), (3) hold. Specifically, assume there is an idempotent $e \in S$, such that

$$\phi_e(S) = X \times G \times Y,$$

and for which the three mentioned properties hold.

Consider $\pi_n = \delta_e * \mu_n * \delta_e$.

Then $\operatorname{supp}(\pi_n) \subset G$ and for $H \subset G$

$$\sum 1 - \pi_n(H) = \sum 1 - \delta_e * \mu_n * \delta_e\big[\{e\} \times H \times \{e\}\big].$$

Notice that

$$\begin{aligned} &\mu_n * \delta_e(X \times H \times \{e\}) \\ &\quad = \sum \mu_n\big[\big(X \times H \times \{e\}\big)y^{-1}\big]\delta_e(y) \\ &\quad = \mu_n\big[\big(X \times H \times \{e\}\big)e^{-1}\big] \\ &\quad = \mu_n(X \times H \times Y). \end{aligned}$$

Similarly, $\pi_n(H) = \mu_n(X \times H \times Y)$.

This implies that

$$\sum 1 - \pi_n(H) < \infty$$

and

$$\sum_n 1 - \pi_n(h_n H_1 h_{n+1}^{-1}) = \infty$$

for all subgroups $H_1 \subset H$ and all sequences $(h_n) \subset G$.

Thus, by the result for finite groups [4],

$$\pi_{k,n} \longrightarrow \rho_k, \quad \rho_k \longrightarrow w_H.$$

Consider $\operatorname{supp}(\mu_1 * \delta_e * \mu_2)$ and its relationship to $\operatorname{supp}(\mu_1 * \mu_2)$.

Let

$$s_1 = (x_1, g_1, y_1) \in \operatorname{supp}(\mu_1),$$

$$s_2 = (x_2, g_2, y_2) \in \operatorname{supp}(\mu_2).$$

Then

$$\begin{aligned} s_1 \cdot s_2 &= (x_1, g_1, y_1)(x_2, g_2, y_2) \\ &= (x_1, g_1 e g_2, y_2) \\ &= (x_1 g_1 y_1)(e, e, e)(x_2, g_2, y_2) \\ &= s_1 e s_2. \end{aligned}$$

Since there is equality throughout we have shown that $s_1 s_2 = s$ iff $s_1 e s_2 = s$.

Thus

$$\begin{aligned} \mu_1 * \delta_e * \mu_2(s) &= \sum_{s_1 s_2 s_3 = s} \mu_1(s_1)\delta_e(s_2)\mu_2(s_3) \\ &= \sum_{s_1 e s_3 = s} \mu_1(s_1)\mu_2(s_3) \\ &= \sum_{s_1 s_3 = s} \mu_1(s_1)\mu_2(s_3) \\ &= \mu_1 * \mu_2(s). \end{aligned}$$

Now let ν_∞ be any tail idempotent of (μ_n) with

$$\mu_{k,n_i} \to \nu_k, \quad \nu_{n_i} \to \nu_\infty.$$

By the above,

$$\begin{aligned} \pi_{k,n_i} &= \delta_e * \mu_{k+1} * \delta_e \cdots * \delta_e * \mu_{n_i} * \delta_e \\ &= \delta_e * \mu_{k,n_i} * \delta_e. \end{aligned}$$

Letting $n_i \to \infty$,

$$\rho_k = \delta_e * \nu_k * \delta_e.$$

Choose $k = n_i$ and letting $n_i \to \infty$

$$w_H = \delta_e * \nu_\infty * \delta_e.$$

Thus $e \operatorname{supp}(\nu_\infty) e = H$ which shows that $e \in \operatorname{supp}(\nu_\infty)$. Therefore,

$$\nu_\infty = \alpha * w_H * \beta$$

where $\alpha \in P(X)$, $\beta \in P(Y)$.

Now let (m_i) be such that

$$\mu_{k,m_i} \to \nu'_k$$

$$\nu'_{m_i} \to \nu'_\infty = (\nu'_\infty)^2$$

$$\nu_{m_i} \to \lambda.$$

Then by convolution equations shown in [2, pg. 292], we have that

$$\mu_{k,n_i} = \mu_{k,m} * \mu_{m,n_i}$$

$$\nu_k = \mu_{k,m} * \nu_m.$$

Thus,

$$\begin{aligned} \nu_k &= \nu'_k * \lambda \qquad\qquad \text{(listing } m = m_i\text{)} \\ &= \nu'_k * \nu'_\infty * \lambda * \nu_\infty \\ &= \nu'_k. \end{aligned}$$

Therefore, $\nu_k = \nu'_k$ for all k, and from this it follows that $\mu_{k,n} \to \nu_k$ for all k as $n \to \infty$. □

4. An Example

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite semigroup and $\mathcal{A}(S)$ the real semigroup algebra of S. Let $\mathcal{A}_1(S) \subset \mathcal{A}(S)$ be the subset

$$\mathcal{A}_1(S) = \left\{ \sum_{i=1}^{n} p_i s_i \mid p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1 \right\}.$$

Notice $\mathcal{A}_1(S)$ is closed under multiplication. With each element $a = \sum_{i=1}^{n} p_i s_i$ of $\mathcal{A}_1(S)$ associate the probability measure $\mu \in \mathcal{P}(S)$ where $\mu(s_i) = p_i$. If $a = \mathcal{A}_1(S)$ and $\mu \in \mathcal{P}(S)$ have the above relationship, we will write $a \sim \mu$. It is an elementary exercise to show that if $a_1 \sim \mu_1$ and $a_2 \sim \mu_2$, then $a_1 \cdot a_2 \sim \mu_1 * \mu_2$. This notation provides a simple way to write examples for small matrix semigroups.

Now consider $T_k = p_1^k s_1 + p_2^k s_2 + p_3^k s_3 + p_4^k s_4$ where

$$s_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$s_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad s_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The four matrices $S = \{s_1, s_2, s_3, s_4\}$ form a rectangular group. Choosing $e = s_1$ to form the Rees product provides $\phi_e(S) = X \times G \times Y$ where

$$X = \{s_1\}, \qquad G = \{s_1, s_2\}, \qquad Y = \{s_1, s_3\},$$

the principal left ideals of S are $Ss_1 = \{s_1, s_2\}$ and $Ss_3 = \{s_3, s_4\}$, and the only subgroup of G is $H = \{s_1\}$. Thus the conditions of Theorem 3 are satisfied if

(1) $\lim_{k\to\infty} p_1^k + p_2^k$ exists (this forces $\lim_{k\to\infty} p_3^k + p_4^k$ to exist),
(2) $p_1^k + p_3^k \geq \frac{1}{k}$ eventually,
(3) $p_2^k + p_4^k \geq \frac{1}{k}$ eventually.

Under these conditions, $\mu_{k,n}$ converges for all $k \geq 0$.

Notice that

$$T_k = \begin{bmatrix} p_1^k & p_3^k & p_2^k & p_4^k \\ p_1^k & p_3^k & p_2^k & p_4^k \\ p_2^k & p_4^k & p_1^k & p_3^k \\ p_2^k & p_4^k & p_1^k & p_3^k \end{bmatrix}$$

is a sequence of stochastic matrices. It is hoped that this connection can be exploited in later work to find conditions for the convergence of non-homogeneous Markov chains.

5. Future Directions

A recent paper by Högnäs and Mukherjea [6] provides some hope that these results can be extended to the case where the sandwich function is non-trivial.

If the general finite completely simple case can be solved, it will be straightforward to extend it to all finite semigroups with some mild conditions to ensure the limit measures have their support in the completely simple minimal ideal. Work on this aspect of the problem continues.

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