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The Structure of Limit Measures and their Supports on Topological Semigroups

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ABSTRACT. Given a sequence of measures (μ_n) on a topological semigroup S, a measure λ on S is called a tail limit of (μ_n) if for some subsequence of integers (n_i), $\mu_{k,n_i} = \mu_{k+1} * \cdots * \mu_{n_i}$ converges weakly to ν_k for all k and λ is a weak limit point of the sequence (ν_k). The main theorem of this paper characterizes the supports of the tail limits on compact completely simple semigroups. Some applications of the theorem and various open problems are discussed.

1. Introduction

Let (X_n) be a sequence of independent random variables taking values in a topological semigroup S. Let $(\mu_n) \subset$ Prob (S) be the sequence of associated probability distributions. That is, for any Borel set $B \subset S$

$$P(X_n \in B) = \mu_n(B).$$

We wish to address the following question: Under what conditions will the associated random walk

$$X_{k,n} = X_{k+1} \cdot X_{k+2} \ldots \cdot X_n$$

converge in distribution for all $k \geq 0$?

Since the X_i's are assumed independent, this becomes a question concerning the (weak) convergence of the convolution product

$$\mu_{k,n} = \mu_{k+1} * \mu_{k+2} * \cdots * \mu_n$$

where for $\mu, \nu \in$Prob(S),

$$\mu * \nu(B) = \int \mu(Bx^{-1})\nu\,(dx) = \int \nu(x^{-1}B)\mu(dx),$$

$$Bx^{-1} = \left\{ y \,\middle|\, yx \in B \right\}, \quad \text{and} \quad x^{-1}B = \left\{ y \,\middle|\, xy \in B \right\}$$

For $\mu \in$ Prob(S), we define the support of μ to be

$$S_\mu := \left\{ x \,\middle|\, \mu(N(x)) > 0 \text{ for all open } N(x) \text{ containing } x \right\}$$

Part of the interest stems from an application of the theory to discrete dynamical systems. One of the main problems in dynamical systems is the characterization of the attractor of the system.

Example . As above, let (X_n) be a sequence of independent random variables taking values in the semigroup of nonnegative $n \times n$ matrices. Let $W_n = X_n \cdot \ldots \cdot X_1$, and for $x \in (\mathbf{R}^n)^+$, consider the random orbit $(W_n(x))$. Then the attractor $A(x)$ of x is defined as

$$A(x) = \left\{ y \in (\mathbf{R}^n)^+ \,\middle|\, P(W_n(x) \in N(y) \text{ infinitely often }) > 0 \right\}$$

where $N(y)$ is an arbitrary open neighborhood of y. If (X_n) is identically distributed with distribution μ, then Mukherjea has shown [**4**] that under very general condions, there is a beautiful relationship between the attractor of x and the minimal ideal I of the closed semigroup generated by S_μ. Namely, the attractor is the image of x under I. That is,

$$A(x) = \left\{ Mx \,\middle|\, M \in I \right\}.$$

Thus a careful analysis of random walks on topological semigroups should shed a great deal of light on the theory of discrete stochastic dynamical systems.

The techniques used in this paper are those of algebraic probability theory. When S is a compact semigroup, then $(\mathrm{Prob}\,(S), *)$ is a compact semigroup, and its algebraic structure can be exploited to prove the desired convergence theorems.

The organization of this paper will be as follows. After this introduction, Section 2 will collect certain preliminary results and specify our notation. Section 3 will contain the main theorems concerning the structure of the supports of the limit measures, and in Section 4 new convergence results will be proven using these structure theorems. Finally in Section 5, we will discuss some related open problems.

2. Preliminaries

Throughout we will assume that S is a compact, second countable, Hausdorff semigroup. We write $E(S)$ to denote the set of idempotents in S. Since $\mathrm{Prob}\,(S)$ is compact, given any sequence of integers (p_i) there exists a subsequence $(n_i) \subset (p_i)$ such that

$$\begin{aligned} \mu_{k,n_i} &\to \nu_k \quad \text{for all} \quad k \\ \nu_{n_i} &\to \nu_\infty = \nu_\infty^2 \\ \nu_k &= \nu_k * \nu_\infty \end{aligned} \tag{1}$$

The arrow denotes weak convergence. In the following, when $\nu_k (\equiv \nu_k(n_i))$ is written, it will mean that (1) holds. The measure ν_∞ will be referred to as a *tail limit.*

Now consider an arbitrary limit point λ of (ν_k). Then for some sequence (m_i), $\nu_{m_i} \to \lambda$. Consider (m_i) in the *original* products. Let $\nu'_k = \nu'_k(m_i)$ and $\nu'_{m_i} \to \nu'_\infty = (\nu'_\infty)^2$. Now let λ' be such that $\nu'_{n_i} \to \lambda'$. The following semigroup equations in $\mathrm{Prob}\,(S)$ were demonstrated in [**1**].

$$\begin{aligned} \nu_k &= \nu'_k * \lambda & \nu'_k &= \nu_k * \lambda' \\ \lambda &= \lambda * \nu_\infty & \lambda' &= \lambda' * \nu'_\infty \\ \lambda' &= \nu_\infty * \lambda' & \lambda &= \nu'_\infty * \lambda \\ \nu_\infty &= \lambda' * \lambda & \nu'_\infty &= \lambda * \lambda' \end{aligned} \tag{2}$$

These equations indicate an algebraic structure on the tail limits. Indeed, the following result was proved in [**2**].

Theorem 1. *Let $(\mu_n) \subset \mathrm{Prob}(S)$, where S is a compact semigroup. Suppose (μ_n) has a unique tail idempotent. Let*

$$V = \left\{ (\nu_k) \,\middle|\, \mu_{k,n_i} \to \nu_k \textit{ for some } (n_i) \right\}$$

Let $\mathcal{F}$ be the set of tail limits of (μ_n). That is,

$$\mathcal{F} = \left\{ \lambda \,\middle|\, \lambda \textit{ is a weak limit point for some } (\nu_k) \in V \right\}$$

Finally, let $G =< \mathcal{F} >$, the semigroup generated by $\mathcal{F}$. Then G is a group. ∎

Remark . If S is abelian, then from the convolution equations,

$$\nu_\infty = \lambda' * \lambda = \lambda * \lambda' = \nu'_\infty$$

and thus (μ_n) has a unique tail idempotent.

Since $\mu * \nu(B) = \int \mu(Bx^{-1})\nu\,(dx)$ it can easily be shown that $S_{\mu*\nu} = \overline{S_\mu \cdot S_\nu}$. Thus the convolution equations (2) imply the following identities on the supports:

$$\begin{aligned} S_\lambda &= \overline{S_\lambda \cdot S_{\nu_\infty}} & \qquad S_{\lambda'} &= \overline{S_{\lambda'} \cdot S_{\nu'_\infty}} \\ S_{\lambda'} &= \overline{S_{\nu_\infty} \cdot S_{\lambda'}} & S_\lambda &= \overline{S_{\nu'_\infty} \cdot S_\lambda} \\ S_{\nu_\infty} &= \overline{S_{\lambda'} \cdot S_\lambda} & S_{\nu'_\infty} &= \overline{S_\lambda \cdot S_{\lambda'}} \end{aligned} \tag{3}$$

In the current context, S_μ is a completely simple semigroup whenever μ is idempotent [**5**, Theorem 3.17]; these semigroups will play a central role in our development.

Let S be completely simple. For each $e \in E(S)$, we define an isomorphism Φ_e by

$$\Phi_e(s) = (E(sSe),\, ese,\, E(eSs)).$$

By Rees' theorem we have a representation $\Phi_e(S) = X \times G \times Y \cong XGY = S$, where $X = E(Se)$, $G = eSe$, and $Y = E(eS)$. Given $s = (x, g, y) \in \Phi_e(S)$, define $\Pi_X(s) = x$, $\Pi_G(s) = g$, and $\Pi_Y(s) = y$.

Though quite routine, the following result is included for convenience.

Proposition 2. *Let $S \cong X \times G \times Y$ be a compact completely simple semigroup, $e \in E(S)$, $X = E(Se)$, $G = eSe$, $Y = E(eS)$.*

(i) If H is a subgroup of S, then H is contained in a maximal subgroup $\widehat{G}$ of S of the form

$$\widehat{G} \cong \{x\} \times G \times \{y\} \cong fSf, \textit{ where } f \in E(S).$$

Moreover, there exists a subgroup $\widehat{H}$ of G such that

$$H \cong \{x\} \times \widehat{H}(yx)^{-1} \times \{y\}$$

(ii) If S' is any subsemigroup of S, then S' is completely simple. Thus S' is a disjoint union of isomorphic subgroups H_α of S':

$$S' = \bigcup_\alpha H_\alpha \cong \bigcup_{\alpha,\beta} \left[\{x_\alpha\} \times \widehat{H}(y_\beta x_\alpha)^{-1} \times \{y_\beta\} \right]$$

(iii) Let S/ϕ_H be the quotient semigroup generated by the H-classes of S and let $\psi : S \to S/\phi_H$ be the canonical homomorphism. Then $\psi(S')$ is a subsemigroup of S/ϕ_H and $\psi(S') = \cup_\alpha \psi(H_\alpha)$.

∎

3. Structure Theorems

The objects of investigation in this section are the supports of tail limit measures. In an earlier paper [**6**], Mukherjea made the observation that when S is a finite group, the supports of the tail idempotents were conjugate subgroups of S. Our next theorem shows that this observation can be generalized considerably. For the following, ν_∞, ν'_∞ are as in (3) above.

Theorem 3. *Let S_{ν_∞} be a subsemigroup of the completely simple semigroup S. Let $e \in E(S_{\nu_\infty})$ be a primitive idempotent, and set $\Phi = \Phi_e$. Write $\Phi(S) = X \times G \times Y$ and $\Phi(S_{\nu_\infty}) = X_1 \times H_1 \times Y_1$. Then H_1 is a subgroup of G, and if H_2 is the subgroup of G such that*

$$\Phi(S_{\nu'_\infty}) = \bigcup_{\alpha,\beta}\Big[\{x_\alpha\} \times H_2(y_\beta x_\alpha)^{-1} \times \{y_\beta\}\Big], \tag{4}$$

then H_1 and H_2 are conjugate.

Proof. Note that $(YX) \subset G$ and $(Y_1X_1) \subset H_1$. Also, $H_1 = eS_{\nu_\infty}e \subset eSe = G$, so H_1 is a subgroup of G.
Next we show that

$$\Phi(S_\lambda) = \overline{\bigcup\Big[\{x\} \times gH_1 \times Y_1\Big]} \tag{5}$$

where the union is taken over all triples $(x, g, y) \in \Phi(S_\lambda)$. Since Φ is an isomorphism, the equations (3) imply

$$\Phi(S_\lambda) = \overline{\Phi(S_\lambda) \cdot \Phi(S_{\nu_\infty})} = \overline{\bigcup_{(x,g,y)\in\Phi(S_\lambda)}\Big[(x, g, y) \cdot (X_1 \times H_1 \times Y_1)\Big]}$$

Let $s \in S_\lambda$, and write $\Phi(s) = (a, b, c)$. For some triple $(x, g, y) \in \Phi(S_\lambda)$,

$$(a, b, c) \in \overline{(x, g, y) \cdot (X_1 \times H_1 \times Y_1)}.$$

Since $\overline{(x, g, y) \cdot (X_1 \times H_1 \times Y_1)} = \overline{\{x\} \times g(yX_1)H_1 \times Y_1}$, we deduce that $a = x$, $c \in Y_1$, and thus also $y \in Y_1$. It then follows that $yX_1 \subset H_1$. Thus

$$\Phi(S_\lambda) = \overline{\bigcup\Big[\{x\} \times g(yX_1)H_1 \times Y_1\Big]} = \overline{\bigcup\Big[\{x\} \times gH_1 \times Y_1\Big]}$$

A similar argument proves that

$$\Phi(S_{\lambda'}) = \overline{\bigcup\Big[X_1 \times H_1g' \times \{y'\}\Big]} \tag{6}$$

where the union is taken over all triples $(x', g', y') \in \Phi(S_{\lambda'})$. From (5) and (6) it immediately follows that

$$\Phi(S_{\nu'_\infty}) = \overline{\Phi(S_\lambda) \cdot \Phi(S_{\lambda'})} = \overline{\bigcup_{(x,g,y)}\bigcup_{(x',g',y')}\Big[\{x\} \times gH_1g' \times \{y'\}\Big]} \tag{7}$$

Let $X_2 = \Pi_X[\Phi(S_{\nu'_\infty})]$, $Y_2 = \Pi_Y[\Phi(S_{\nu'_\infty})]$. Then we claim that

$$\Pi_X[\Phi(S_\lambda)] = X_2 = \Pi_X[\Phi(S_{\nu'_\infty})] \quad \text{and} \quad \Pi_Y[\Phi(S_{\lambda'})] = Y_2 = \Pi_Y[\Phi(S_{\nu'_\infty})].$$

To see this, note that as a consequence of Proposition 2, part (iii),

$$(s,t) \in X_2 \times Y_2 \Longrightarrow \{s\} \times H_2(ts)^{-1} \times \{t\} \subset \Phi(S_{\nu'_\infty}).$$

The assertion then follows from (5), (6) and (7) above.

Since $\Phi(S_{\nu_\infty}) = \overline{\Phi(S_{\lambda'}) \cdot \Phi(S_\lambda)}$, we have for all $(x,g,y) \in \Phi(S_\lambda)$ and all $(x',g',y') \in \Phi(S_{\lambda'})$,

$$\Big[X_1 \times H_1 g' \times \{y'\}\Big] \cdot \Big[\{x\} \times gH_1 \times Y_1\Big] \subset X_1 \times H_1 \times Y_1.$$

Thus

$$H_1 g'(y'x)gH_1 = H_1. \tag{8}$$

It immediately follows that $Y_2X_2 \subset \bigcup\Big[(g')^{-1}H_1g^{-1}\Big]$, where $g \in \Pi_G[\Phi(S_\lambda)]$ and $g' \in \Pi_G[\Phi(S_{\lambda'})]$.

Now fix $x \in X_2$. Then by (8),

$$(x,g,y), \text{ and } (x,\widehat{g},\widehat{y}) \in \Phi(S_\lambda) \Longrightarrow gH_1 = \widehat{g}H_1 = \psi_1(x)H_1,$$

where $\psi_1 : X_2 \to G$. Similarly,

$$(x',g',y') \in \Phi(S_{\lambda'}) \Longrightarrow g' \in H_1\psi_2(y'),$$

where $\psi_2 : Y_2 \to G$.

By (4) and (6), we may write

$$\overline{\bigcup_{(x,g,y)} \bigcup_{(x',g',y')} \Big[\{x\} \times gH_1g' \times \{y'\}\Big]} = \overline{\bigcup_{\alpha,\beta}\Big[\{x_\alpha\} \times H_2(y_\beta x_\alpha)^{-1} \times \{y_\beta\}\Big]}$$

We may also use $S_\lambda = \overline{S_{\nu'_\infty} \cdot S_\lambda}$ to deduce that

$$\bigcup\Big[\{x\} \times gH_1 \times Y_1\Big] = \Big[\bigcup_{\alpha,\beta}(\{x_\alpha\} \times H_2(y_\beta x_\alpha)^{-1} \times \{y_\beta\})\Big] \cdot \Big[\bigcup(\{\widehat{x}\} \times \widehat{g}H_1 \times Y_1)\Big]$$

Since $\Pi_X[\Phi(S_\lambda)] = \Pi_X[\Phi(S_{\nu'_\infty})] = X_2$, there exists $s \in X_2$ such that for some $t \in Y_2$,

$$H_2(ts)^{-1}\widehat{g}(ts)H = gH_1, \quad \text{and} \quad g, \widehat{g} \in \Pi_G[\Phi(S_\lambda)].$$

Thus $H_2\widehat{g}H_1 = gH_1$ and in particular $H_2\widehat{g} \subset gH_1$, whence $H_2 \subset gH_1(\widehat{g})^{-1}$. Observing that $\widehat{g} \in gH_1$, and writing $\widehat{g} = gh$ for some $h \in H_1$, so that $(\widehat{g})^{-1} = h^{-1}g^{-1}$, we get

$$H_2 \subset gH_1g^{-1} \tag{9}$$

Now since $S_{\nu'_\infty} = \overline{S_\lambda \cdot S_{\lambda'}}$ we can write

$$\bigcup_{\alpha,\beta}\Big[\{x_\alpha\} \times H_2(y_\beta x_\alpha)^{-1} \times \{y_\beta\}\Big] = \overline{\Big[\bigcup(\{x\} \times gH_1 \times Y_1)\Big]\Big[\bigcup(X_1 \times H_1g' \times \{y'\})\Big]}$$

If (x_α, y_β) is chosen appropriately, we deduce that $gH_1g' \subset H_2(y'x)^{-1}$. Thus $gH_1g'(y'x) \subset H_2$. By (8) this is equivalent to

$$gH_1g^{-1} \subset H_2 \tag{10}$$

Combining (9) and (10), we have that H_1 and H_2 are conjugate, as required.

■

Note that we have also proved the following.

Theorem 6. *Let S be a compact completely simple semigroup and let λ, λ', ν_∞, ν'_∞ be as above. Let $e \in E(S_{\nu_\infty})$ and let $\Phi = \Phi_e$. Set $\Phi(S_{\nu_\infty}) = X_1 \times H_1 \times Y_1$ and let*

$$\Phi(S_{\nu'_\infty}) = \bigcup_{\alpha,\beta}\Big[\{x_\alpha\} \times H_2(y_\beta x_\alpha)^{-1} \times \{y_\beta\}\Big]$$

Then the following are true:

(i) $\Phi(S_\lambda) = \bigcup_\alpha \Big[\{x_\alpha\} \times \psi_1(x_\alpha)H_1 \times Y_1\Big] = \bigcup_\alpha \Big[\{x_\alpha\} \times H_2\psi_1(x_\alpha) \times Y_1\Big]$
where $\psi_1 : X_1 \to G$.

(ii) $\Phi(S_{\lambda'}) = \bigcup_\beta \Big[X_1 \times H_1\psi_2(y_\beta) \times \{y_\beta\}\Big]$
where $\psi_2 : Y_2 \to G$

(iii) $\Pi_X[\Phi(S_\lambda)] = \Pi_X[\Phi(S_{\nu'_\infty})]$; $\Pi_Y[\Phi(S_{\lambda'})] = \Pi_Y[\Phi(S_{\nu'_\infty})]$ ■

In the case where S_{ν_∞} and $S_{\nu_{\infty'}}$ share a common idempotent, the theorem takes the following simpler form:

Corollary 7. *Let $S \cong X \times G \times Y$ be a compact completely simple semigroup and let λ, λ', ν_∞, ν'_∞ be as above. Let $e = e^2 \in S_{\nu_\infty} \cap S_{\nu'_\infty}$, and $\Phi = \Phi_e$. Set*

$$\Phi(S_{\nu_\infty}) = X_1 \times H_1 \times Y_1$$
$$\Phi(S_{\nu_{\infty'}}) = X_2 \times H_2 \times Y_2$$

Then the following are true:

(i) H_1, H_2 *are conjugate subgroups of* G.

(ii) $\Phi[S_\lambda] = X_2 \times hH_1 \times Y_1 = X_2 \times H_2 h \times Y_1$
for some $h \in \Pi_G[\Phi(S_\lambda)]$.

(iii) $\Phi[S_{\lambda'}] = X_1 \times H_1 h' \times Y_2 = X_1 \times h'H_2 \times Y_2$
for some $h' \in \Pi_G[\Phi(S_{\lambda'})]$. ■

4. Applications

In this section, the structure of the support of tail limits will figure prominently. The conditions in the theorems allow the semigroup properties of the probability measures to be used to full advantage in proving the convergence theorems. The first result is a simple, but extremely useful lemma.

Lemma 8. *Let S be a completely simple semigroup. Let $\mu, \nu \in$ Prob (S). Suppose $L \subset S$ is a left ideal and $R \subset S$ is a right ideal. Then the following are true.*

(i) $\mu * \nu(L) = \nu(L)$.

(ii) $\mu * \nu(R) = \mu(R)$.

Proof. Notice that

$$Lz^{-1} = \begin{cases} S & \text{if } z \in L \\ \emptyset & \text{otherwise} \end{cases}$$

Thus $\mu * \nu(L) = \int_L \mu(Lz^{-1})\nu(dz) = \nu(L)$. Similarly,

$$z^{-1}R = \begin{cases} S & \text{if } z \in R \\ \emptyset & \text{otherwise} \end{cases}$$

and so $\mu * \nu(R) = \int_R \nu(z^{-1}R)\mu(dz) = \mu(R)$. ∎

When a sequence (μ_n) has a unique tail idempotent, then from Theorem 1, the tail limits generate a group of measures. As indicated earlier, if S is abelian, the existence of a unique tail limit is immediate. However, as the next theorem shows, this uniqueness property can occur in the non-abelian case as well.

Theorem 9. *Let $(\mu_n) \subset$ Prob(S), where $S = XGY \cong X \times G \times Y$ is a compact completely simple semigroup and G is abelian. Suppose that for arbitrary closed subsets $C_1 \subset X$, $C_2 \subset Y$, and weak limit points μ', μ'' of (μ_n), we have*

$$\mu'(C_1 S) = \mu''(C_1 S) \quad \textit{and} \quad \mu'(SC_1) = \mu''(SC_2)$$

Then (μ_n) has a unique idempotent.

Proof. Suppose $\nu_k = \nu_k(n_i)$, $\nu'_k = \nu'_k(p_i)$ and let $\mu_{n_i} \to \mu'_1, \mu_{p_i} \to \mu''_1$. Then for some $\beta_k = \beta_k(n_i - 1), \gamma_k = \gamma_k(p_i - 1)$,

$$\nu_k = \beta_k * \mu'_1$$
$$\nu'_k = \nu_k * \mu''_1$$

Thus if ν_∞, ν'_∞ are the respective tail idempotents associated with (ν_k) and (ν'_k), and β, γ are limit points of (β_k) and (γ_k), then

$$\nu_\infty = \beta * \mu'_1 \tag{11}$$
$$\nu'_\infty = \gamma * \mu''_1$$

Similarly,

$$\nu_\infty = \mu'_2 * \alpha \tag{12}$$
$$\nu'_\infty = \mu''_2 * \pi$$

where μ_2' and μ'' are limit points of (μ_n). Using (11), (12), and Lemma 8, and observing that C_1S and SC_2 are right and left ideals, respectively, we see that

$$\begin{aligned}\nu_\infty(C_1S) &= \nu'_\infty(C_1S) \\ \nu'_\infty(SC_2) &= \nu'_\infty(SC_2)\end{aligned} \tag{13}$$

By Theorem 3,

$$\Phi(S_{\nu_\infty}) = X_1 \times H_1 \times Y_1$$
$$\text{and} \quad \Phi(S_{\nu'_\infty}) = \bigcup_i \Big[\{x_\alpha\} \times H_2(y_\beta x_\alpha)^{-1} \times \{y_\beta\}\Big]$$

where H_1, H_2 are conjugate. Since G is abelian, $H_1 = H_2$. Using (13) it is easy to see that

$$\Pi_X(\Phi(S_{\nu'_\infty})) = \Pi_X(\Phi(S_{\nu_\infty})) = X_1$$
$$\Pi_Y(\Phi(S_{\nu'_\infty})) = \Pi_Y(\Phi(S_{\nu_\infty})) = Y_1$$

Then $(y_\beta x_\alpha) \in H_1$ and $\Phi(S_{\nu'_\infty}) = X_1 \times H_1 \times Y_1$. Since each of ν_∞, ν'_∞ are idempotent,

$$\nu_\infty = \lambda_1 * \omega_{H_1} * \lambda_2$$
$$\nu'_\infty = \lambda'_1 * \omega_{H_1} * \lambda'_2$$

where $\lambda_1, \lambda'_1 \in$Prob (X), $\lambda_2, \lambda'_2 \in$Prob (Y), and ω_{H_1} is the Haar measure on H_1 [**5**, Theorem 3.17].

Also, if $A \subset X$, $B \subset G$, and $C \subset Y$, then

$$\nu_\infty(A \times B \times C) = \lambda_1(A)\omega_H(B)\lambda_2(C) \tag{14}$$

Similarly for ν'_∞ (see [**5**] for details).

Now since $\Phi(C_1S) = C_1 \times G \times Y$ and $\Phi(SC_2) = X \times G \times C_2$, (13) and (14) combine to yield $\lambda_1(C_1) = \lambda'_1(C_1)$ and $\lambda_2(C_2) = \lambda'_2(C_2)$. Thus $\nu_\infty = \nu'_\infty$. ■

The next theorem is a generalization of Theorem 3 in [**7**]. The assumption of a unique tail idempotent takes the place of the abelian assumption in the earlier work. It is relevant to note that the same theorem can be proven for general compact groups without this additional assumption.

Theorem 10. *Let S be a compact semigroup and $(\mu_n) \subset$ Prob(S). Suppose (μ_n) has a unique tail idempotent, ν_∞. Then, $\mu_{k,n} \to \nu_k$ for all k iff there exists a closed subsemigroup $S' \subset S$ such that the following conditions hold:*

(i) For all $\varepsilon > 0$, there exists k_0 and a sequence n_k such that for $k \geq k_0$, $n \geq n_k$ and any open set $U \supset S'$,

$$\mu_{k,n}(U) > 1 - \varepsilon$$

(ii) For all proper closed semigroups $S'' \subset S'$ there exists $\delta > 0$, K_0, (n_k) and an open set $V \supset S''$ such that for all $k \geq K_0$, $n \geq n_k$

$$\mu_{k,n}(V) < 1 - \delta$$

Proof. Suppose properties (i) and (ii) hold for some S', and let $\nu_k \equiv \nu_k(n_i)$. Then for some $(p_i) \subset (n_i)$, $\mu_{p_i,p_{i+1}} \to \nu_\infty$.

Using a proof similar to that of Theorem 3 in [**7**], it can be shown that $S_{\nu_\infty} = S'$.

Now let $\lambda \neq \nu_\infty$ be any tail limit of (ν_k) and consider S_λ. Suppose there exists $x \in S_\lambda$ such that $x \notin S'$. Then there exists $U(x)$, $U(S')$ such that $U(x) \cap U(S') = \emptyset$. Also, since $\nu_{q_i} \to \lambda$, given $\varepsilon > 0$, there exists I such that $i \geq I$ implies $\nu_{q_i}(U(x)) > \varepsilon$. But $\mu_{q_i,p_i} \to \nu_{q_i}$ as $p_i \to \infty$. Hence there exists P such that $p_j \geq P \Rightarrow \mu_{q_i,p_j}(U(x)) > \varepsilon$. Thus $\mu_{q_i,p_j}(U(S')) < 1-\varepsilon$. But this is a contradiction. Therefore, $S_\lambda \subset S_{\nu_\infty}$.

Again by [**5**, Theorem 3.17], since ν_∞ is an idempotent probability measure it has a completely simple support and can be factored as follows:

$$\begin{aligned} &\nu_\infty = \lambda_1 * \omega_H * \lambda_2, \\ \text{with}\quad &S_{\lambda_1} = X; \quad S_{\lambda_2} = Y \\ \text{and}\quad &S_{\nu_\infty} = X \times H \times Y. \end{aligned}$$

Now consider $\nu_\infty * \lambda * \nu_\infty$. Note that $S_\lambda \subset S_{\nu_\infty}$ and $Y S_{\nu_\infty} X = H$, which is a group. Moreover,

$$\begin{aligned} \nu_\infty * \lambda * \nu_\infty &= \lambda_1 * (\omega_H * \lambda_2 * \lambda * \lambda_1 * \omega_H) * \lambda_2 \\ &= \lambda_1 * \omega_H * \lambda_2 \\ &= \nu_\infty, \end{aligned}$$

since $S_{\lambda_2} \cdot S_\lambda \cdot S_{\lambda_1} \subset H$.

Let $\nu'_k = \nu'_k(q_i)$ be arbitrary and let λ be such that $\nu_{q_i} \to \lambda$. Then $\nu_k = \nu'_k * \lambda$ by the convolution equations (2). By Theorem 1, since ν_∞ is an identity for λ,

$$\begin{aligned} \nu_k &= \nu_k * \nu_\infty \\ &= (\nu'_k * \lambda) * \nu_\infty \\ &= \nu'_k * (\nu_\infty * \lambda * \nu_\infty) \\ &= \nu'_k * \nu_\infty \\ &= \nu'_k. \end{aligned}$$

Thus $\mu_{k,n}$ converges for all k.

Now suppose that for all $k \geq 1$, $\mu_{k,n} \to \nu_k$, where (ν_k) has a unique tail idempotent ν_∞. By [**2**, Theorem 1] this implies that $\nu_k \to \nu_\infty$.

Let $S' = S_{\nu_\infty}$. Then if $U \supset S'$ is any open set, for all $\varepsilon > 0$, there exists $K_{(\varepsilon)}$ such that $\nu_k(U) > 1 - \varepsilon$ for all $k \geq K_{(\varepsilon)}$. Thus S' clearly satisfies property (i).

Now suppose $S'' \subset S'$ is a proper closed subsemigroup. As before, we can find $U(x), V(S'')$ such that $U(x) \cap V(S'') = \emptyset$. Now there exists $\delta > 0$ such that $\nu_\infty(U(x)) > \delta$. Also there exists K_0 such that $k \geq K_0 \Rightarrow \nu_k(U(x)) > \delta$. Since $\mu_{k,n} \to \nu_k$, there exists $N_{(k)}$ such that

$$\mu_{k,n}(U(x)) > \delta \quad \text{for} \quad n > N_{(k)} \Rightarrow \mu_{k,n}(V) < 1 - \varepsilon.$$

Thus property (ii) follows.

∎

In the next result we assume that the supports of the tail idempotents have at least one element in common.

Theorem 11. *Let $S \cong X \times G \times Y$ be a compact completely simple semigroup. Let $(\mu_n) \subset \operatorname{Prob}(S)$. Suppose*

$$\bigcap_{\nu \in E(\mathcal{F})} S_\nu \neq \emptyset.$$

Then $\mu_{k,n} \to \nu_k$ for all $k \geq 0$ iff the following are true

(i) $E(\mathcal{F}) = \mathcal{F}$.

(ii) For any closed subset $C \subset Y$ and any two weak limit points, μ', μ'' of (μ_n)

$$\mu'(SC) = \mu''(SC)$$

Proof. Suppose $\mu_{k,n} \to \nu_k$ for all $k \geq 0$. If λ is any weak limit point of (ν_k), then by [**1**, Theorem 1], λ is idempotent. Now let μ' be any weak limit point of (μ_n). Then for some (p_i), $\mu_{p_i} \to \mu'$, and consequently

$$\mu_{k,p_i} = \mu_{k,p_i - 1} * \mu_{p_i} \quad \Rightarrow \quad \nu_k = \beta_k * \mu'$$

where β_k is some weak limit of μ_{k,p_i-1}. Notice that SC is a left ideal of S and thus by Lemma 8,

$$\nu_k(SC) = \int \beta_k(SCy^{-1})\mu'(dy) = \mu'(SC). \tag{15}$$

But μ' was an arbitrary weak limit point of (μ_n) and so

$$\mu'(SC) = \mu''(SC) = \nu_k(SC)$$

for all $k \geq 0$, and for any two weak limits μ', μ''.

To prove the converse, assume all tail limits are idempotent. Let

$$\nu_k = \nu_k(n_i), \quad \nu_{n_i} \to \nu_\infty = \nu_\infty^2$$
$$\nu'_k = \nu'_k(m_i), \quad \nu'_{m_i} \to \nu'_\infty = (\nu'_\infty)^2$$

Let λ be such that $\nu_{m_i} \to \lambda$. Then by assumption $\lambda = \lambda^2$. By Corollary 7,

$$S_\lambda = X_2 \times H_2 h \times Y_1 = X_2 \times hH_1 \times Y_1$$

where $S_{\nu_\infty} = X_1 \times H_1 \times Y_1$ and $S_{\nu'_\infty} = X_2 \times H_2 \times Y_2$. However, since λ is idempotent, there is a subgroup $H_3 \subset G$ such that $H_2 h = hH_1 = H_3$. Thus $h^{-1} \in H_2 \cap H_1$, and $H_1 = H_2 = H_3 = H$.

Now, since ν_∞, ν'_∞, and λ are all idempotents, they can be expressed as follows:

$$\nu_\infty = \mu_1 * \omega_H * \nu_1$$
$$\nu'_\infty = \mu_2 * \omega_H * \nu_2$$
$$\lambda = \mu_3 * \omega_H * \nu_3$$

Now $\nu_\infty(SC) = \nu'_\infty(SC)$ by (11) and (12) in the proof of Theorem 9. Moreover,

$$\mu_1 * \omega_H * \nu_1(SC) = \mu_1 * \omega_H * \nu_2(SC) \implies \nu_1(C) = \nu_2(C)$$

where C is an arbitrary closed subset of $Y = E(eS)$. It follows that $\nu_1 = \nu_2 = \nu$ (say). Using the convolution identities (2)

$$\begin{aligned}
\nu_k &= \nu_k' * \lambda \\
&= \nu_k' * (\nu_\infty' * \lambda * \nu_\infty) \\
&= \nu_k' * (\mu_2 * \omega_H * \nu) * (\mu_3 * \omega_H * \nu_3) * (\mu_1 * \omega_H * \nu) \\
&= \nu_k' * (\mu_2 * \omega_H * \nu) \\
&= \nu_k' * \nu_\infty' \\
&= \nu_k' \quad (\text{since } S_\nu \cdot S_{\mu_3} \text{ and } S_{\nu_3} \cdot S_{\mu_1} \quad \text{are subsets of} \quad H)
\end{aligned}$$

Thus $\mu_{k,n} \to \nu_k$ weakly for all $k \geq 0$.

■

Notice that if S is a compact group, then $e \in \bigcap_{\nu \in E(\mathcal{F})} S_\nu$, where e is the identity of S. Also, $SC = S$ for each closed $C \subset Y$. Thus we have the following corollary (which to the best of our knowledge was previously unknown).

Corollary 12. *Let S be a compact group. Let $(\mu_n) \subset \mathrm{Prob}(S)$. Then $\mu_{k,n} \to \nu_k$ for all $k \geq 0$ if and only if $E(\mathcal{F}) = \mathcal{F}$.*

■

5. Some Open Problems

Many interesting problems in this area remain unsolved. Two that are particularly relevant in this context are the following.

(1) Let $S_1 = \overline{\langle \mathcal{F} \rangle}$ where $\mathcal{F}$ is the set of tail limits of some sequence of measures $(\mu_n) \subset \mathrm{Prob}\,(S)$, where S is not abelian. Then S_1 is a compact subsemigroup of $\mathrm{Prob}(S)$. What is the minimal ideal of S_1?

(2) Notice that, in general, for λ, λ' as before

$$\lambda' * \lambda * \lambda' = \lambda' * \nu_\infty' = \lambda'$$
$$\lambda * \lambda' * \lambda = \lambda * \nu_\infty = \lambda$$

then all of the tail limits are regular elements. Is S_1 a regular semigroup?

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