

The Road Coloring Problem

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Abstract. The synchronizing word of deterministic automaton is a word in the alphabet of colors (considered as letters) of its edges that maps the automaton to a single state. A coloring of edges of a directed graph is synchronizing if the coloring turns the graph into deterministic finite automaton possessing a synchronizing word.

The road coloring problem is a problem of synchronizing coloring of directed finite strongly connected graph with constant outdegree of all its vertices if the greatest common divisor of lengths of all its cycles is one. The problem was posed by Adler, Goodwyn and Weiss over 30 years ago and evoked a noticeable interest among the specialists in theory of graphs, deterministic automata and symbolic dynamics.

The positive solution of the problem is presented.

Keywords: graph, deterministic finite automata, synchronizing word, road coloring problem.

Introduction

The road coloring problem originates in [2] and stated explicitly in [1] for strongly connected directed finite graph with constant outdegree of all its vertices where the greatest common divisor (gcd) of lengths of all its cycles is one. The edges of the graph are unlabeled. The task is to find a labelling of the edges that turns the graph into a deterministic finite automaton possessing a synchronizing word. So the road coloring problem is connected with the problem of existence of synchronizing word for deterministic complete finite automaton.

The condition on gcd is necessary [1], [5]. It can be replaced by equivalent property that there exists a partition of the set of vertices on subsets $V_1, V_2, \dots, V_k = V_1$ such that every edge with begin in V_i has its end in V_{i+1} [5], [10]. The outdegree of the vertex can be considered also as the size of an alphabet where the letters denote colors.

The road coloring problem is important in automata theory: a synchronizing coloring makes the behavior of an automaton resistant against input errors since, after detection of an error, a synchronizing word can reset the automaton back to its original state, as if no error had occurred. The problem is also essential for symbolic dynamics and appeared first just in papers from this area.

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Together with the Černy conjecture [12], [13] the road coloring problem belongs to the most fascinating problems in the theory of finite automata. However, at the same time it was considered as "notorious open problem" [9].

For some positive results in the considered area see [3], [4], [6], [7], [8], [10], [11], a detailed history of investigations one can find in [4]. The concept from [6] of the weight of a vertex and the concept of a stable pair of states from [5], [8] with corresponding results and consequences is essentially used below.

We prove that the road coloring problem has positive solution. So finite directed strongly connected graph with constant outdegree of all vertices and with gcd of the lengths of all its cycles equal to one has synchronizing coloring.

Preliminaries

The finite directed strongly connected graph with constant outdegree of all its vertices where the gcd of lengths of all its cycles is one will be called *AGW graph* as aroused by Adler, Goodwyn and Weiss.

Let u be left eigenvector with positive components having no common divisor of adjacency matrix of a graph with vertices $\mathbf{p}_1, \dots, \mathbf{p}_n$. The i -th component u_i of the vector u let us call *the weight* of the vertex \mathbf{p}_i and denote by $w(\mathbf{p}_i)$. The sum of the weights of the vertices from a set D let us denote by $w(D)$ and call *the weight* of D .

$|P|$ - the size of the set of states of the automaton (of vertices of the graph) P . If there exists a path in automaton from the state \mathbf{p} to the state \mathbf{q} and the edges of the path are consecutively labeled by $\sigma_1, \dots, \sigma_k$ then for $s = \sigma_1 \dots \sigma_k \in \Sigma^+$ let us write $\mathbf{q} = \mathbf{p}s$.

Let P_s be the map of the set of states P by help of $s \in \Sigma^+$ and let P_s^{-1} be the maximal set of states Q such that $Qs = P$.

A word $s \in \Sigma^+$ is called *synchronizing* word of the transition graph Γ of the automaton if $|\Gamma s| = 1$.

A coloring of a directed finite graph is *synchronizing* if the coloring turns the graph into a deterministic finite automaton possessing a synchronizing word.

A pair of states \mathbf{p}, \mathbf{q} (of vertices of the transition graph) of an automaton will be called *synchronizing* if $\mathbf{p}s = \mathbf{q}s$ for some $s \in \Sigma^+$. In opposite case, if for any s $\mathbf{p}s \neq \mathbf{q}s$ let us call the pair *deadlock*.

A synchronizing pair of states \mathbf{p}, \mathbf{q} (of vertices of the transition graph) of an automaton is called *stable* if for any word u there exists a word v such that $\mathbf{p}uv = \mathbf{q}uv$ [5], [8].

The set of all outgoing edges of a vertex let us call a *bunch* if these edges are also ingoing edges of one vertex.

The subset D of states (of vertices of the transition graph) of an automaton such that $w(D)$ is maximal and $|Ds| = 1$ for some word s let us call *F-maximal* as supposed by Friedman [6].

The subset Γs of states (of vertices of the transition graph) such that every pair of states from the set is deadlock let us call *F-clique*.

1 Some properties of F -clique and of coloring free of stable pairs

The road coloring problem was formulated for AGW graphs [1] and only such graphs are considered below. We exclude from the consideration also the primitive cases of graphs with loops and of only one color [1], [10]. Let us formulate two important results from [6] and [8] in the following form:

Theorem 1 [6] *For every coloring of AGW graph Γ , for any word t and F -maximal set F the set Ft^{-1} is also F -maximal.*

There exists a partition of Γ on F -maximal sets of the same weight.

Some side condition on AGW graph declared in [6] are not used in the proof of this statement.

Theorem 2 [8] *Suppose $\mathbf{p} \rho \mathbf{q}$ if the pair of vertices \mathbf{p}, \mathbf{q} from AGW graph Γ is stable via some coloring of Γ .*

Then ρ is congruence relation, Γ/ρ is AGW graph and synchronizing coloring of Γ/ρ implies synchronizing coloring of Γ .

Lemma 1 *Let W be the sum of the weights of all vertices of the AGW graph Γ and w be the sum of the weights of all vertices of its F -maximal set via some coloring. Then the size of any F -clique of the coloring is the same and equal to W/w .*

Proof. Two states from F -clique could not belong to one F -maximal set because this pair is not synchronizing. By Theorem 1 there exists a partition of Γ on F -maximal sets of weight w . So the partition consists from W/w F -maximal sets and to every F -maximal set belongs at most one state from F -clique. Consequently the size of any F -clique is not greater than W/w .

Let $\Gamma s = \{\mathbf{q}_i\}$ be F -clique. The weight of the set $\mathbf{q}_i s^{-1}$ for every i is not greater than w . The sum of the weights for all i is the weight W of Γ . The size of F -clique is not greater than W/w . Therefore $\mathbf{q}_i s^{-1}$ is F -maximal set of weight w for every i and the size of F -clique is W/w .

Lemma 2 *For any word s and F -clique F via some coloring of AGW graph Γ the set Fs is also F -clique and any state [vertex] \mathbf{p} belongs to some F -clique. Let a word s synchronize some subset of vertices R from Γ . Then R is a subset of some F -maximal set.*

Proof. Any pair \mathbf{p}, \mathbf{q} from F -clique F is a deadlock, therefore the pair $\mathbf{p}s, \mathbf{q}s$ from Fs also is the deadlock. For any \mathbf{r} from strongly connected graph Γ there exists a word s such that $\mathbf{r} = \mathbf{p}s$, whence \mathbf{r} belongs to F -clique Fs .

The vertex Rs as well as any other vertex belongs to some F -maximal set U . Therefore Us^{-1} in view of the theorem 1 is also F -maximal set. The set R is a subset of Us^{-1} .

Lemma 3 *Let A and B ($|A| > 1$) be distinct F -cliques via some coloring without stable pairs of the AGW graph Γ .*

Then $|A| - |A \cap B| = |B| - |A \cap B| > 1$.

Proof. Let us suppose the contrary: $|A| - |A \cap B| = 1$. By Lemma 1 $|A| = |B|$. So $|B| - |A \cap B| = 1$ too. The pair of states $\mathbf{p} \in A \setminus B$ and $\mathbf{q} \in B \setminus A$ is not stable. Therefore for some word s the pair $(\mathbf{p}s, \mathbf{q}s)$ is deadlock. Any pair of states from F -clique A and from F -clique B as well as from F -cliques As and Bs is deadlock. So any pair of states from the set $(A \cup B)s$ is deadlock. One has $|(A \cup B)s| = |A| + 1 > |A|$.

In view of the theorem 1 there exists a partition of size $|A|$ (Lemma 1) of Γ on F -maximal sets. To every F -maximal set belongs at most one state from $(A \cup B)s$ because every pair of states from this set is deadlock and no deadlock could belong to F -maximal set. It contradicts to the fact that the size of $(A \cup B)s$ is greater than $|A|$.

Lemma 4 *Let some coloring of AGW graph Γ have no stable couples. Then any vertex of Γ has at most one ingoing bunch.*

Proof. If a vertex \mathbf{p} has two ingoing bunches from \mathbf{q} and \mathbf{r} then the couple \mathbf{q}, \mathbf{r} is stable for any coloring because $\mathbf{q}\alpha = \mathbf{r}\alpha = \mathbf{p}$ for any $\alpha \in A^+$.

2 The spanning set of cycles and trees with maximal number of edges in the cycles

Définition 1 *Let us call a subgraph S of the AGW graph Γ a spanning set of Γ if to S belong all vertices of Γ and exactly one outgoing edge of any vertex. A maximal tree of the spanning set S having no common edges with cycles from S let us call a tree of S .*

The minimal length of path from vertex \mathbf{p} through edges of the spanning set S to the root of its tree on cycle from S let us call the level of \mathbf{p} in S .

Remark 1 *Any spanning set S consists of disjoint cycles and trees with roots on cycles, any tree and cycle of S is defined identically, the level of the vertex from cycle is zero, the vertices of trees except root have positive level, the vertex of maximal positive level has no ingoing edge from S .*

Lemma 5 *Let H be a set of vertices of level $n > 0$ from some tree of the spanning set S of AGW graph Γ . Then via coloring of Γ such that all edges of S have the same color α , for any F -clique F holds $|F \cap H| \leq 1$.*

Proof. Some power of α synchronizes all states of given level of the tree and maps them on the root. Any couple from F -clique could not be synchronized and therefore could not belong to H .

Lemma 6 *Let AGW graph Γ have a spanning set R of only disjoint cycles (without trees).*

Then Γ has also another spanning set with exactly one vertex of maximal positive level.

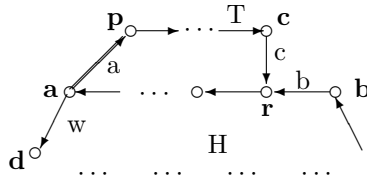
Proof. The spanning set R has only cycles and therefore the levels of all vertices are equal to zero. In view of $\gcd = 1$ in strongly connected graph Γ not all edges belong to bunch. Therefore there exist two edges $u = \mathbf{p} \rightarrow \mathbf{q} \notin R$ and $v = \mathbf{p} \rightarrow \mathbf{s} \in R$ with common beginning but such that $\mathbf{q} \neq \mathbf{s}$. Let us replace the edge $v = \mathbf{p} \rightarrow \mathbf{s}$ from R by u . Then only the vertex \mathbf{s} has maximal level $L > 0$ in new spanning set.

Lemma 7 *Let any coloring of AGW graph Γ be free of stable couples. Then Γ has a spanning set R such that all its vertices of maximal positive level belong to one non-trivial tree of R .*

Let us consider R with maximal number of vertices [edges] in its cycles. In view of Lemma 6, suppose that R has non-trivial trees and let $L > 0$ be maximal value of the level of a vertex.

Further consideration is necessary only if at least two vertices of level L belong to distinct trees of R with distinct roots.

Let us consider a tree T from R with vertex \mathbf{p} of maximal level L and edge c from vertex \mathbf{c} to the tree root $\mathbf{r} \in T$ on the path of length L from \mathbf{p} . Let the root \mathbf{r} belong to the cycle H of R with the edge $b = \mathbf{b} \rightarrow \mathbf{r} \in H$. There exists also an edge $a = \mathbf{a} \rightarrow \mathbf{p}$ that does not belong to R because Γ is strongly connected and \mathbf{p} has no incoming edge from R .



Let us extend the path from \mathbf{p} to \mathbf{r} of maximal length L in T by one or two from three following ways:

- A) replace the edge from R with beginning in the vertex \mathbf{a} by the edge $a = \mathbf{a} \rightarrow \mathbf{p}$,
- B) replace the edge b from R by some another outgoing edge of the vertex \mathbf{b} ,
- C) replace the edge c from R by some another outgoing edge of the vertex \mathbf{c} .

Our aim is to extend the maximal level of vertex on the extension of the tree T widely than the level of vertex in other trees from R . Let us begin from

A) If the vertex \mathbf{a} does not belong to the cycle H then the length of the new path from \mathbf{a} to \mathbf{r} through \mathbf{p} in new spanning set is $L + 1$ and the vertex \mathbf{r} is a root of new tree containing all vertices of maximal level (\mathbf{a} or its ancestors).

So let us assume $w = \mathbf{a} \rightarrow \mathbf{d} \in H$. In this case the vertices \mathbf{p} , \mathbf{r} and \mathbf{a} belong to a cycle H_1 with new edge a of a new spanning set R_1 . The cycle H_1 from R_1 replaced in R the cycle H . If the length of path from \mathbf{r} to \mathbf{a} in H is r_1 then H_1 has length $L + r_1 + 1$. From the cycle H remains in R_1 a path to \mathbf{r} from the vertex \mathbf{d} . Suppose its length is r_2 . So the length of the cycle H is $r_1 + r_2 + 1$. The length of the cycle H_1 is not greater than the length of H because the spanning set R has maximal number of edges in its cycles. So $r_1 + r_2 + 1 \geq L + r_1 + 1$, whence $r_2 \geq L$. If $r_2 > L$ then the length r_2 of the path from \mathbf{d} to \mathbf{r} in a tree of R_1 (and the level of \mathbf{d}) is greater than L and the level of \mathbf{d} (or of ancestor of \mathbf{d})

in tree from R) is desired unique maximal level.

So assume for the further consideration $L = r_2$ and $w = \mathbf{a} \rightarrow \mathbf{d} \in H$.

B) Suppose the set of outgoing edges of the vertex \mathbf{b} is not a bunch. Let us replace in R the edge b by an edge $u = \mathbf{b} \rightarrow \mathbf{u}$ such that $\mathbf{u} \neq \mathbf{r}$. The vertex \mathbf{u} could not belong to the tree T because one has in this case a cycle with all vertices from H and some vertices of T whence its length is greater than $|H|$ and so a new spanning set has number of vertices in its cycles greater than in spanning set R in spite of the choice of R .

If the vertex \mathbf{u} does not belong to T then the tree T is a part of new tree with new root and the path from \mathbf{p} to new root is extended at least by a part of H from the former root \mathbf{r} . The new level L_1 of \mathbf{p} therefore is maximal and greater than the level of any vertex in some another tree.

So for the further consideration suppose that the set of outgoing edges of the vertex \mathbf{b} is a bunch to \mathbf{r} .

C) By Lemma 4, the set of outgoing edges of the vertex \mathbf{c} is not a bunch to \mathbf{r} because \mathbf{r} has another bunch from \mathbf{b} . So one can replace in R the edge c by an edge v to a vertex $\mathbf{v} \neq \mathbf{r}$.

The vertex \mathbf{v} could not belong to T because in this case a new cycle is added to R and therefore a new spanning set has number of vertices in cycles greater than in R .

If the vertex \mathbf{v} belongs to another trees of R and not to cycles then T is a part of new tree T_1 with new root of new spanning set R_1 and the path from \mathbf{p} to new root is extended. So only the tree T_1 has a state of maximal level.

If \mathbf{v} belongs to some cycle $H_1 \neq H$ from R then let us together with replacing c by v replace also the edge w by a . So we extend the path from \mathbf{p} to the new root \mathbf{v} at least by edge $a = \mathbf{a} \rightarrow \mathbf{p}$ and almost all edges of H . Therefore new maximal level $L_1 > L$ has either the vertex \mathbf{d} or its ancestors in a tree from R . Now remains only the case \mathbf{v} belongs to the cycle H . The vertex \mathbf{p} also has level L in new tree T_1 with root \mathbf{v} . The only difference between T and T_1 (as well as between R and R_1) is the root and the ingoing edge of the root. The new spanning set R_1 has maximal number of vertices in cycles like R .

One can repeat the step A) for spanning set R_1 , vertex \mathbf{p} , tree T_1 with root \mathbf{v} , consider the vertices \mathbf{a} and \mathbf{d} and for the length r_3 of path from \mathbf{d} to new root \mathbf{v} in cycle H obtain $L = r_3$. From $\mathbf{v} \neq \mathbf{r}$ follows $r_3 \neq r_2$, though $L = r_3$ and $L = r_2$.

Thus anyway exists a spanning set with vertices of maximal level in one non-trivial tree.

Theorem 3 *Let any coloring of AGW graph Γ be free of stable couples. Then Γ has synchronizing coloring.*

Proof. By Lemma 7, Γ has spanning set R such that the vertices of maximal level belong to one non-trivial tree.

Let us give to the edges of R the color α and denote by C the set of all vertices from monochromatic cycles from R colored by α . Then let us color the remaining edges of Γ by another colors arbitrarily. Let us denote by T the vertices of unique tree with vertices of maximal level $L > 0$.

By Lemma 2, in strongly connected graph Γ for every word s and F -clique F of size $|F| > 1$ the set Fs also is F -clique and there exists F -clique F' such that F' has non-empty intersection with any non-empty set of vertices.

In particular, some F has non-empty intersection with the set of vertices of maximal level L . By Lemma 5 this intersection has only one vertex. The word α^{L-1} maps therefore F on F -clique F_1 of size $|F|$ such that $|F_1 \setminus C| = 1$ because the sequence of edges of color α from any tree of R leads to the root of the tree, the root belongs to a cycle colored by α from C and T has the most long such sequence. So $|F_1 \setminus C| = 1$ and $|C \cap F_1| = |F_1| - 1 = |F| - 1$.

Let the integer m be a common multiple of the lengths of all considered cycles from C colored by α . So for any \mathbf{p} from C as well as from $F_1 \cap C$ holds $\mathbf{p}\alpha^m = \mathbf{p}$. Therefore for F -clique $F_2 = F_1\alpha^m$ holds $F_2 \subseteq C$ and $C \cap F_1 = F_1 \cap F_2$.

Thus two F -cliques F_1 and F_2 of size $|F| > 1$ have $|F| - 1$ common vertices. So $|F_1 \setminus (F_1 \cap F_2)| = 1$ in spite of the Lemma 3.

Consequently $|F| = 1$ and the considered coloring is synchronizing.

Theorem 4 *Every AGW graph Γ has synchronizing coloring.*

The proof follows from Theorems 3 and 2.

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