ASYMPTOTIC STABILITY AND BOUNDEDNESS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study a system $(D) x' = F(t, x_t)$ of functional differential equations, together with a scalar equation $(S) x' = -a(t)f(x) + b(t)g(x(t-h))$ as well as perturbed forms. A Liapunov functional is constructed which has a derivative of a nature that has been widely discussed in the literature. On the basis of this example we prove results for (D) on asymptotic stability and equi-boundedness.

1991 *Mathematics Subject Classification.* 34K20.

Key words and phrases. Functional differential equations, asymptotic stability, boundedness, Liapunov functionals.

*On leave from Northeast Normal University, Jilin, Changchun, PRC. That research is partially supported by NNSF, # 19331061.

1. Introduction. In this paper we discuss asymptotic stability and boundedness of solutions of a system of functional differential equations

$$x' = F(t, x_t)$$

by means of Liapunov functionals. The conditions are motivated by a specific Liapunov functional for the scalar equation

$$x' = -a(t)f(x) + b(t)g(x(t-h)) + p(t).$$

That Liapunov functional has the basic form of one which was studied by Krasovskii [17; pp. 143–160] and which has been studied intensively up to the present time. It is that functional which we focus on here.

But there is another form for the derivative of a Liapunov functional,

$$V' \leq -\delta|F(t, x_t)| + M,$$

which has been studied since the 1960's, with several recent contributions. We show that our Liapunov functional also satisfies that type of condition.

Thus, we obtain new results for the scalar equation, extend the Krasovskii theorem, and provide a strong example of current interest.

2. Asymptotic stability. We begin with the scalar equation

$$x'(t) = -a(t)f(x(t)) + b(t)g(x(t-h)) \tag{1}$$

in which it is assumed that all functions are continuous, that $h > 0$, and that there are positive constants α and β with

$$xf(x) > 0, \quad |f(x)| \geq |g(x)| \text{ for } 0 < |x| \leq \beta, \tag{2}$$

$$-a(t) + |b(t+h)| \leq -\alpha a(t), \quad a(t) \geq 0, \tag{3}$$

and

$$\int_0^\infty a(t)\,dt = \infty. \tag{4}$$

We now show that these conditions suffice to prove that the zero solution of (1) is asymptotically stable. The reader may consult Yoshizawa [24; pp. 183–213] (or any book on functional differential equations and Liapunov's direct method) for definitions of stability and for properties of Liapunov functionals. Some properties are formalized later.

As a motivation for the conditions of our first result we define a standard Liapunov functional for (1) and arrive at a nonstandard conclusion. The reader may be interested in noting that nothing is said about boundedness of $a(t)$ or $b(t)$; we believe this is entirely new for the general form of (1).

For a solution $x(t)$ of (1) we define

$$V(t, x_t) = |x(t)| + \int_{t-h}^{t} |b(s+h)|\,|g(x(s))|\,ds \tag{5}$$

so that if we write $V(t) = V(t, x_t)$ we have

$$\begin{aligned} V'(t) &\le -a(t)|f(x)| + |b(t)g(x(t-h))| \\ &\quad + |b(t+h)g(x)| - |b(t)g(x(t-h))| \\ &\le [-a(t) + |b(t+h)|]\,|f(x)| \end{aligned}$$

or by (3)

$$V'(t) \le -\alpha a(t)|f(x)| \le -\alpha a(t)|g(x)|. \tag{6}$$

Now from (2) and (3) we have

$$|b(t+h)g(x)| \le |b(t+h)f(x)| \le (1-\alpha)a(t)|f(x)|$$

so that from (5) we obtain

$$|x(t)| \le V(t, x_t) \le |x(t)| + (1-\alpha)\int_{t-h}^{t} a(s)|f(x(s))|\,ds. \tag{7}$$

It is well-known (cf. Krasovskii [17; p. 144]) that (6) and (7) imply that the zero solution of (1) is stable.

But the simplicity of these relations immediately implies that all solutions tend to zero. Indeed, from (6) we have

$$0 \le V(t) \le V(t_0) - \alpha \int_{t_0}^{t} a(s)|f(x(s))|\,ds$$

so that the integral converges; hence, in (7) we see that $\int_{t-h}^{t} a(s)|f(x(s))|\,ds \to 0$ as $t \to \infty$. But by (4) we apply (2) and conclude that there is a sequence $\{t_n\} \to \infty$ such that $x(t_n) \to 0$. Thus, $V(t_n) \to 0$; but $V'(t) \le 0$ so for $t \ge t_n$ we have

$$|x(t)| \le V(t) \le V(t_n) \to 0,$$

as required.

This will motivate our first theorem and it is these sorts of relations on which this paper focuses. However, in recent years there has been renewed interest in relations on Liapunov functionals which are very different from those in (6) and (7). It is very simple at this point to illustrate such a relation using (1) – (4).

The idea begins with a system

$$x' = h(x) \tag{0}$$

and a Liapunov function $V(x)$. If $x(t)$ is a solution of (0), then

$$\begin{aligned} V'_{(0)}(x(t)) &= \operatorname{grad} V \cdot h = \\ &\qquad |\operatorname{grad} V|\,|h| \cos\theta, \end{aligned}$$

If V is carefully chosen, we may find $\delta > 0$ with

$$V'_{(0)}(x(t)) \le -\delta |x'|.$$

Thus, V is bounded by the arc length of a solution. Generalizations of this idea were discussed by Becker-Burton-Zhang [1], Burton [2–5], Burton-Casal-Somolinos [8], Haddock

[13], Erhart [12] some time ago. Recently, Burton-Hering [10], Burton-Makay[11], Makay [18], Kobayashi [16], and Tsuruta [19] have resumed the investigation. Thus, it seems worth while to state a strong example of that sort since it can be done with economy in view of (5) and (6).

Let (2), (3), and (4) hold and perturb (1) to

$$x' = -a(t)f(x) + b(t)g(x(t-h)) + p(t) \tag{1*}$$

where p is continuous, $|p(t)| \le M$ for some $M > 0$. Let $k > 1$ and define

$$V(t, x_t) = |x(t)| + k\int_{t-h}^{t} |b(s+h)|\,|g(x(s))|\,ds \tag{5*}$$

so that if $x(t)$ is a solution of (1*) and if we write $V(t) = V(t, x_t)$ then we have

$$V'(t) \le [-a(t) + k|b(t+h)|]\,|f(x)| - (k-1)|b(t)|\,|g(x(t-h))| + |p(t)|.$$

By (3), there is a $k > 1$, $d > 0$, and $r > 0$ with

$$V'(t) \le -d[|x'(t)| + a(t)|f(x(t))|] + rM. \tag{6*}$$

The aforementioned references give many results on boundedness and stability from relations like (6*) without asking boundedness of $|x'(t)|$.

Our work here develops (5) and (6); we say no more about (6*). A general theorem will now be formulated.

Let $(C, \|\cdot\|)$ be the Banach space of continuous functions $\varphi : [-h, 0] \to R^n$ with the supremum norm, $h > 0$, and for $A > 0$ if $x : [-h, A) \to R^n$ is continuous then define $x_t \in C$ by $x_t(s) = x(t+s)$ for $-h \le s \le 0$. If $\delta > 0$, then C_δ is the δ-ball in C.

Let $F : [0, \infty) \times C_\beta \to R^n$ be continuous, take bounded sets into bounded sets, let $F(t, 0) = 0$, and let $\beta > 0$. Then

$$x' = F(t, x_t) \tag{8}$$

is a system of functional differential equations with finite delay. If $\varphi \in C_\beta$ and $t_0 \ge 0$, then there is a solution $x(t) = x(t, t_0, \varphi)$ of (8) on a maximal interval $[t_0, \gamma)$ with $\gamma = \infty$ or $\limsup_{t \to \gamma^-} |x(t)| = \beta$, and $x_{t_0} = \varphi$.

In this paper we employ continuous functions $W_i : [0, \infty) \to [0, \infty)$ which are strictly increasing, satisfy $W_i(0) = 0$, and are called wedges.

Let $\||\cdot\||$ denote the L^2-norm on C. Krasovskii [17; p. 155] showed that if there is a continuous function $V : [0, \infty) \times C_\beta \to [0, \infty)$ and wedges W_i with

$$W_1(|\varphi(0)|) \le V(t, \varphi) \le W_2(|\varphi(0)|) + W_3(\||\, \varphi\||) \tag{9}$$

and

$$V'_{(8)}(t, x_t) \le -W_4(|x(t)|) \tag{10}$$

then $x = 0$ is asymptotically stable. In [6] we showed that the conclusion is actually uniform asymptotic stability. Wang [20] showed that $\||\cdot\||$ could be replaced by any L^p-norm, while other improvements were made by Burton-Hatvani [9], Burton-Hering [10], Burton-Makay[11], Hatvani [17], Ko [15], Wang [21–23], Zhang [25–26], and others. But it seems that all of these asked that $V(t, \varphi) \le W(\|\varphi\|)$, at least on a sequence of intervals.

Def. 1. The zero solution of (8) is said to be stable if for each $\varepsilon > 0$ and $t_0 \ge 0$ there exists $\delta > 0$ such that $[\varphi \in C_\delta, t \ge t_0]$ imply that $|x(t, t_0, \varphi)| < \varepsilon$. It is asymptotically stable if it is stable and if for each $t_0 \ge 0$ there exists $\eta > 0$ such that $\varphi \in C_\eta$ implies that $x(t, t_0, \varphi) \to 0$ as $t \to \infty$.

Our foregoing work with (1) suggests and motivates the following result.

Theorem 1. *Let V, $H : [0, \infty) \times C_\beta \to [0, \infty)$ be continuous, $H(t, 0) = 0$, and W_i be wedges with*

(i) $W_1(|\varphi(0)|) \le V(t, \varphi) \le W_2(|\varphi(0)|) + W_3\left(\int_{t-h}^{t} H(s, \varphi)\, ds\right)$,

(ii) $V'_{(8)}(t, x_t) \le -W_4(H(t, x_t))$,

(iii) *W_4 is convex downward,*

(iv) if $\varepsilon > 0$ and $x : [t_0, \infty) \to R^n$, $\varepsilon \le |x(t)| \le \beta$, then $\int_{t_0}^{\infty} W_4(H(s, x_s))\, ds = \infty$.

Under these conditions the zero solution of (8) is asymptotically stable.

Proof. Conditions (i) and (ii) are well-known to imply stability. Let $x(t)$ be a solution of (8) on $[t_0, \infty)$ with $|x(t)| < \beta$. From (ii) we have for $V(t) = V(t, x_t)$ that $0 \le V(t) \le V(t_0) - \int_{t_0}^{t} W_4(H(s, x_s))\, ds$; but by (iv) there is a sequence $\{t_n\} \uparrow \infty$ with $x(t_n) \to 0$. By renaming, let $t_{n+1} - t_n \ge h$. Notice that $\int_{t_n - h}^{t_n} H(s, x_s)\, ds \to 0$ as $n \to \infty$; for if there is an $\varepsilon > 0$ and a subsequence $\{t_{n_k}\}$ with

$$\int_{t_{n_k}-h}^{t_{n_k}} H(s, x_s)\, ds \ge \varepsilon,$$

then $t \ge t_{n_K}$ implies (by Jensen's inequality) that

$$\begin{aligned} 0 \le V(t) &\le V(t_0) - \sum_{k=1}^{K} h W_4\left(\frac{1}{h}\int_{t_{n_k}-h}^{t_{n_k}} H(s, x_s)\, ds\right) \\ &\le V(t_0) - \sum_{k=1}^{K} h W_4(\varepsilon/h) \end{aligned}$$

so $V(t) \to -\infty$ as $K \to \infty$, a contradiction. In particular, we can find $\{t_n\} \uparrow \infty$ with $x(t_n) \to 0$ and $\int_{t_n - h}^{t_n} H(s, x_s)\, ds \to 0$ so $V(t_n) \to 0$. Then $V' \le 0$ and so $t \ge t_n$ implies that

$$W_1(|x(t)|) \le V(t, x_t) \le V(t_n, x_{t_n}) \to 0$$

as $t \to \infty$. This completes the proof. □

Def. 2. The zero solution of (8) is said to be equi-asymptotically stable if it is stable and if for each $t_0 \ge 0$ and $\mu > 0$ there exist $\delta > 0$ and $T > 0$ such that $[\varphi \in C_\delta, t \ge t_0 + T]$ imply that $|x(t, t_0, \varphi)| < \mu$.

Theorem 2. *Let (i), (ii), (iii) of Theorem 1 hold and suppose there is a wedge W_5 and a continuous function $S(t)$ with*

(v) $\int_{t-h}^{t} H(s, \varphi)\, ds \le W_5(\|\varphi\|)S(t)$ whenever $\varphi \in C_\beta$.

If, in addition, for each $\varepsilon > 0$ there is a $\delta > 0$ such that $x : [t_0, \infty) \to R^n$ continuous and *(vi)* $\varepsilon \le |x(t)| \le \beta$ *imply that* $\int_{t-h}^{t} W_4(H(s,x_s))\,ds \ge \delta$,

then $x = 0$ *is equi-asymptotically stable.*

Proof. It is still true that $x = 0$ is stable. Let $t_0 \ge 0$ and $\mu > 0$ be given. For this t_0 and $\beta > 0$, find δ_1 of stability. We will find $T > 0$ such that $\varphi \in C_{\delta_1}$ and $t \ge t_0 + T$ imply that $|x(t, t_0, \varphi)| < \mu$.

Let $\varphi \in C_{\delta_1}$ be arbitrary and $x(t) = x(t, t_0, \varphi)$. Consider the intervals

$$I_n = [t_0 + (n-1)h, t_0 + nh], \quad n = 1, 2, 3, \dots.$$

Notice that if there is a $\bar{t} \ge t_0$ with

$$(*) \qquad W_2(|x(\bar{t})|) + W_3\left(\int_{\bar{t}-h}^{\bar{t}} H(s,x_s)\,ds\right) < W_1(\mu)$$

then $|x(t)| < \mu$ for $t \ge \bar{t}$.

Case 1. For a given n, suppose that $W_2(|x(t)|) \ge W_1(\mu)/2$ for each $t \in I_n$. By integration of (ii) and use of (vi) we find $\delta > 0$ with

$$V(t_0 + nh) \le V(t_0 + (n-1)h) - \delta.$$

Case 2. There is a $t_n^* \in I_n$ with $W_2(|x(t_n^*)|) < W_1(\mu)/2$, but

$$W_3\left(\int_{t_n^*-h}^{t_n^*} H(s,x_s)\,ds\right) \ge W_1(\mu)/2.$$

Then by Jensen's inequality, integrating (ii) yields

$$\begin{aligned} V(t_n^*) &\le V(t_n^* - h) - hW_4(W_3^{-1}(W_1(\mu)/2)/h) \\ &=: V(t_n^* - h) - \lambda. \end{aligned}$$

Hence, for a given n either (*) holds or V decreases by an amount

$$r = \min[\delta, \lambda]$$

on every interval $I_{n-1} \cup I_n$. As $V(t_0) \le W_2(\beta) + S(t_0)W_5(\beta)$, there is an N so that $(*)$ holds on some I_n with $n < N$. Thus, $T = Nh$, and the proof is complete. □

3. Boundedness. Notice that if we perturb (1) to

$$x' = -a(t)f(x) + b(t)g(x(t-h)) + p(t) \tag{11}$$

where $p : [0, \infty) \to R$ and there is an $M > 0$ with

$$|p(t)| \le M, \tag{12}$$

then taking the derivative of V defined in (5) along a solution of (11) yields

$$V'_{(11)}(t, x_t) \le -\alpha a(t)|f(x)| + M, \tag{13}$$

while we still have

$$|x(t)| \le V(t, x_t) \le |x(t)| + (1-\alpha)\int_{t-h}^{t} a(s)|f(x(s))|\, ds. \tag{7}$$

We will show that if, in addition to (2) and (3) with $\beta = \infty$, (12), we have $\mu > 0$ and $U > 0$ with

$$\frac{1}{h}\int_{t-h}^{t} \alpha a(s)|f(x)|\, ds \ge 2M + \mu \text{ for } |x| \ge U, \tag{14}$$

then solutions of (11) are equi-ultimately bounded for bound B. This is formulated for (8) when $F(t, 0) = 0$ is removed.

Theorem 3. *In (8) (without $F(t, 0) = 0$) let $\beta = \infty$ and suppose there are continuous functions $V, H : [0, \infty) \times C \to [0, \infty)$, positive constants M, U, and μ, and wedges W_i such that*

(i) $W_1(|\varphi(0)|) \le V(t, \varphi) \le W_2(|\varphi(0)|) + W_3\left(\int_{t-h}^{t} H(s, \varphi)\, ds\right)$

(ii) $V'_{(8)}(t, x_t) \le -W_4(H(t, x_t)) + M$,

(iii) $hW_4\left(\frac{1}{h}\int_{t-h}^{t} H(s, x_s)\, ds\right) \ge 2Mh + \mu$ *whenever x is continuous and $|x(s)| \ge U$ for $t - h \le s \le t$,*

(iv) $W_1(r) \to \infty$ *as* $r \to \infty$, W_4 *is convex downward.*

Then there is a $B > 0$ so that each solution satisfies $|x(t)| < B$ for all large t.

Def. 3. Solutions of (8) are said to be equi-ultimately bounded if there is a $B > 0$ and for any $B_3 > 0$ and $t_0 \geq 0$ there is a $T > 0$ such that $[\varphi \in C_{B_3}, t \geq t_0 + T]$ imply that $|x(t, t_0, \varphi)| < B$; if T is independent of t_0, solutions are uniformly ultimately bounded (UUB).

Def. 4. Solutions of (8) are said to be uniformly bounded (UB) if for each $B_1 > 0$ there is a $B_2 > 0$ such that $[t_0 \geq 0, \varphi \in C_{B_1}, t \geq t_0]$ imply that $|x(t, t_0, \varphi)| < B_2$.

Theorem 4. *If, in addition to the conditions of Theorem 3, there is a continuous function $S(t)$ and wedge W_5 with*

$$\int_{t-h}^{t} H(s, x_s)\, ds \leq S(t) W_5(\|x_t\|)$$

then solutions of (8) are equi-ultimately bounded.

Remark. Theorems 3 and 4 are very unusual since they allow H to be unbounded when x is bounded. It will be much easier to follow the proof after the following results, the first of which generalizes a result in Burton [7] (see Theorem 3).

Theorem 5. *Suppose there is a continuous function $V : [0, \infty) \times C \to [0, \infty)$, positive constants U and M with*

(i) $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3\left(\int_{-h}^{0} W_4(|\varphi(s)|)\, ds\right)$,

(ii) $V'_{(8)}(t, x_t) \leq -W_5(|x(t)|) + M$ *and*

(iii) $W_5(U) > 2M + \frac{1}{h}$, $W_1(r) \to \infty$ *as* $r \to \infty$. *Then solutions are UB and UUB.*

Proof. We first show that Theorem 5 is true provided that we strengthen (ii). Then we show that (ii) can always be strengthened in the required way. Notice that if (iii) holds, then W_4 can always be replaced by a larger W_4, if necessary, so that there is a $U > 0$ with $W_5(W_4(U)) > 2M + \frac{1}{h}$.

Lemma 1. *Suppose there is a continuous function $V : [0, \infty) \times C \to [0, \infty)$, positive constants M, U, and Z with*

(i) $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3\left(\int_{-h}^{0} W_4(|\varphi(s)|)\, ds\right)$,

(ii) $V'_{(8)}(t, x_t) \le -W_5(W_4(|x(t)|)) + M, W_5(W_4(U)) > 2M + \frac{1}{h}$,

$W_5(Z/h) > 2M + \frac{1}{h}$ *and* $W_4(U)h > Z$,

(iii) $W_1(r) \to \infty$ *as* $r \to \infty$, W_5 *convex downward.*

Then solutions of (8) are UB and UUB.

Proof. Let $B_1 > 0$ be given. First, we must find $B_2 > 0$ such that $[t_0 \ge 0, \|\varphi\| \le B_1, t \ge t_0]$ imply that $|x(t, t_0, \varphi)| < B_2$. Then we must find $B > 0$ and for each $B_3 > 0$ find $K > 0$ such that $[t_0 \ge 0, \|\varphi\| \le B_3, t \ge t_0 + K]$ imply that $|x(t, t_0, \varphi)| > B$.

Note from (ii) that

$$(I) \qquad W_5(Z/h) > 2M + \frac{1}{h} \text{ and } W_4(U)h > Z.$$

For an arbitrary $t_0 \ge 0$ and an arbitrary $\varphi \in C_{B_1}$, let $x(t) = x(t, t_0, \varphi), V(t) = V(t, x_t)$, and $I_n = [t_0 + (n-1)h, t_0 + nh]$, $n = 1, 2, \ldots$.

Notice that

$$(II) \qquad V(t+h) \le V(t) + Mh.$$

Next, notice that if there is an $s_1 \in I_2$ with $\int_{s_1-h}^{s_1} W_4(|x(s)|)\, ds \ge Z$, then by Jensen's inequality and (ii) we have by (I) that

$$V(s_1) - V(s_1 - h) \le -hW_5(\frac{1}{h}Z) + Mh$$
$$\le -2Mh - 1 + Mh \le -Mh - 1.$$

This, and the idea in (II) yields

$$(III) \qquad V(t_0 + 2h) \le V(t_0) - 1 \text{ whenever } s_1 \text{ exists.}$$

If s_1 fails to exist, then for all $s \in I_2$ we have $\int_{s-h}^{s} W_4(|x(u)|)\, du < Z$ so there is an $s_2 \in I_2$ with $|x(s_2)| < U$; otherwise, $|x(s)| \ge U$ on I_2 yields $\int_{t_0+h}^{t_0+2h} W_4(|x(s)|)\, ds \ge hW_4(U) > Z$, a contradiction. Thus, $V(s_2) \le W_2(U) + W_3(Z)$ and

$$(IV) \qquad V(t_0 + 2h) \le W_2(U) + W_3(Z) + Mh.$$

Continuing these arguments on $I_4, I_6, \ldots$ we conclude that either

$$(III_{2n}) \qquad V(t_0 + 2nh) \leq V(t_0) - n$$

or

$$(IV_{2n}) \qquad \begin{aligned} V(t_0 + 2nh) &\leq W_2(U) + W_3(Z) + Mh \\ &=: W_1(B) - 2Mh \end{aligned}$$

which defines B.

If (III_{2n}) holds, then from (i) we have

$$\begin{aligned} W_1(|x(t_0 + 2nh)|) &\leq V(t_0 + 2nh) \leq V(t_0) - n \\ &\leq W_2(B_1) + W_3(hW_4(B_1)) - n. \end{aligned}$$

Hence, there is an $N = N(B_1) \neq N(t_0)$, so that (III_{2N}) fails and (IV_{2N}) holds.

Lemma 2. *If (IV_{2n}) holds, then so does (IV_{2n+2}).*

Proof. Either there is an $s_1 \in I_{2n+2}$ with

$$\int_{s_1 - h}^{s_1} W_4(|x(s)|)\, ds \geq Z \text{ so that } V(s_1) \leq V(s_1 - h) - 2Mh - 1$$

and

$$\begin{aligned} V(t_0 + (2n+2)h) &\leq V(t_0 + 2nh) - 1 \\ &\leq W_1(B) - 2Mh - 1 \end{aligned}$$

as required, or the same argument as previously given yields

$$V(t_0 + (2n+2)h) \leq W_2(U) + W_3(Z) + Mh.$$

This proves Lemma 2. □

It now follows that $W_1(|x(t)|) \le V(t) \le W_1(B)$ for all $t \ge t_0 + 2Nh$. We also see that for $t \ge t_0$,

$$\begin{aligned} W_1(|x(t)|) &\le \max[W_1(B), W_2(B_1) + W_3(hB_1) + 2Mh] \\ &=: W_1(B_2). \end{aligned}$$

Replace B_1 with B_3 to complete the proof of UUB. This will prove Lemma 1.

We now finish the proof of Theorem 5 by showing that there is a wedge W so that $W(V(t))$ will satisfy the conditions of Lemma 1.

If W is any wedge, then $W(V(t))$ satisfies

$$\begin{aligned} W_0(|\varphi(0)|) &:= W(W_1(|\varphi(0)|)) \le W(V(t)) \\ &\le W(W_2(|\varphi(0)|) + W_3\left(\int_{-h}^{0} W_4(|\varphi(s)|)\,ds\right)\Big) \\ &\le 2W(W_2(|\varphi(0)|)) + 2W(W_3\left(\int_{-h}^{0} W_4(|\varphi(s)|)\,ds\right) \\ &=: W_7(|\varphi(0)|) + W_8\left(\int_{-h}^{0} W_4(|\varphi(s)|)\,ds\right). \end{aligned}$$

Next, if $R(t, x_t) := W(V(t, x_t))$ then

$$R'_{(8)}(t, x_t) = W'(V(t, x_t))V'_{(8)}(t, x_t)$$

(and if $|x(t)| \ge U$ then

$$V'(t) \le -W_5(|x(t)|) + M \le -\beta W_5(|x(t)|),$$

for some $\beta > 0$) so the inequality continues and we seek W and W_9 with

$$\begin{aligned} &\le -W'(V(t, x_t))\beta W_5(W_4^{-1}W_4(|x(t)|)) \\ &\le -W'(W_1(|x|))\beta W_5(W_4^{-1}(W_4(|x(t)|))) \\ &\le -W_9(W_4(|x(t)|)) \quad \text{(still for } |x(t)| \ge U) \ . \end{aligned}$$

We make W' so large that W_9 can be chosen as convex downward. Since R' is bounded above for $|x(t)| < U$, we can find $\bar{M} > 0$ with

$$R'(t, x_t) \leq -W_9(W_4(|x(t)|)) + \bar{M}$$

and this now completes the proof of Theorem 5. □

Proof of Theorem 3. Let $\varphi \in C$, $t_0 \geq 0$, $x(t) = x(t, t_0, \varphi)$ and $V(t) = V(t, x_t)$. We will find a B, independent of t_0 and φ, with $|x(t)| < B$ for large t.

The proof will proceed just as in Lemma 1.

Consider the intervals I_n once more. Find $Z > 0$ with

$$(I) \qquad hW_4(Z/h) \geq 2Mh + \mu.$$

As before,

$$(II) \qquad V(t+h) \leq V(t) + Mh.$$

Next, notice that if there is an $s_1 \in I_2$ with $\int_{s_1-h}^{s_1} H(s, x_s)\, ds \geq Z$, then by Jensen's inequality and (ii) we have

$$V(s_1) - V(s, -h) \leq -hW_4(Z/h) \leq -2Mh - \mu + Mh.$$

This and (II) yield

$$(III) \qquad V(t_0 + 2h) \leq V(t_0) - \mu, \text{ whenever } s_1 \text{ exists.}$$

If s_1 fails to exist, then for all $s \in I_2$ we have $\int_{s-h}^{s} H(u, x_u)\, du < Z$ so by (iii) there is an $s_2 \in I_2$ with $|x(s_2)| < U$. Thus, $V(s_1) \leq W_2(U) + W_3(Z)$ and

$$(IV) \qquad V(t_0 + 2h) \leq W_2(U) + W_3(Z) + Mh.$$

Continuing these arguments on I_4, I_6, $\ldots$, we conclude that either

$$(III_{2n}) \qquad V(t_0 + 2nh) \leq V(t_0) - n\mu$$

or

$$(IV_{2n}) \qquad V(t_0+2nh) \le W_2(U)+W_3(Z)+Mh$$
$$=: W_1(B)-2Mh$$

which defines B. As $V(t_0)$ is a number, there is an N such that (III_{2N}) fails and (IV_{2N}) holds and $V(t_0+2Nh) \le W_1(B)-2Mh$.

The argument of Lemma 2 holds. If (IV_{2n}) is satisfied, either there is an $s_1 \in I_{2n+2}$ with

$$\int_{s_1-h}^{s_1} H(s,x_s)\,ds \ge Z \text{ so } V(s_1) \le V(s_1-h)-2Mh-\mu$$

and

$$V(t_0+(2n+2)h) \le V(t_0+2nh)-\mu$$
$$\le W_1(B)-2Mh-\mu \text{ so } (IV_{2n+2}) \text{ holds,}$$

or the same argument as previously given works. This proves Theorem 3. □

The proof of Theorem 4 is almost identical to that of Theorem 3. We use

$$\int_{t-h}^{t} H(s,x_s)\,ds \le S(t)W_5(\|x_t\|)$$

to get the bound on N from (III_{2n}).

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