

The following are a list of integration formulas that you should know. Even when you go through the different methods, you reduce the integrals to one of these:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \frac{dx}{x} = \ln |x| + c$$

$$\int \csc^2 x dx = -\cot x + c$$

$$\int e^x dx = e^x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \csc x \cot x dx = -\csc x + c$$

$$\int \cos x dx = \sin x + c$$

The following are nice to know (make your life easier) integrals:

$$\int \tan x dx = -\ln |\cos x| + c$$

$$\int \csc x dx = -\ln |\csc x + \cot x| + c$$

$$\int \cot x dx = \ln |\sin x| + c$$

$$\int \sec x dx = \ln |\sec x + \tan x| + c$$

If a is constant, $a \neq 0$:

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax + c$$

Hint I: If you think you have an antiderivative, just differentiate and see if you end up with the integrand!

Hint II: The line of attack you should take when trying to integrate should be to try these methods in this order. You will always need to have paper and pencil, the correct method doesn't usually bounce off the page and hit you in the face!

Method 1. *Straight forward integration.* The integral can be reduced to one of the first set of integrals.

Example A.
$$\int (x^2 + 1)(x - 3)dx = \int (x^3 - 3x^2 + x - 3)dx = \frac{x^4}{4} - x^3 + \frac{x^2}{2} - 3x + c$$

Example B.
$$\int \frac{1 - \sin^2 x}{1 + \sin x} dx = \int \frac{(1 - \sin x)(1 + \sin x)}{1 + \sin x} dx = \int (1 - \sin x) dx = x + \cos x + c$$

Problems:

1.
$$\int e^x(1 - e^{-x} \sec^2 x) dx$$

2.
$$\int \frac{x^3 - 2x + 4}{x} dx$$

3.
$$\int \cos x \tan x dx$$

Method 2. *Substitution.* A simple substitution reduces the integral to one of the first set of integrals. In fact, the second set came about by using substitution.

Example C.
$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$
 Let $w = \cos x$ then $dw = -\sin x dx$ so that the integral becomes
$$\int -\frac{dw}{w} = -\ln |w| + c = -\ln |\cos x| + c.$$

Example D.
$$\int x e^{x^2} dx.$$
 Let $w = x^2$ then $dw = 2x dx$ so that the integral becomes
$$\int e^w \frac{dw}{2} = \frac{1}{2} e^w + c = \frac{1}{2} e^{x^2} + c.$$

Hint III: Things to look for: if the integrand involves

$$e^{f(x)}, \text{ trig } (f(x)), \frac{1}{f(x)}, (f(x))^n$$

Let $w = f(x)$. This is not an exclusive list!

Hint IV: When substitution is complete, make sure no x 's appear in the integrand.

Problems:

4. $\int \frac{\cos x}{1 + \sin x} dx$. Let $w = 1 + \sin x$.

5. $\int (3x^2 + x) \cos(2x^3 + x^2 + 4) dx$. Let $w = 2x^3 + x^2 + 4$.

6. $\int \frac{(\ln x)^3}{x} dx$. Let $w = \ln x$.

7. $\int \sec^2 x e^{\tan x} dx$

8. $\int x \sqrt{x + 2} dx$

9. $\int \frac{x + 1}{x^2 + 2x - 5} dx$

Method 3. *Integration by parts.* This method undoes the product rule

$$\int u dv = uv - \int v du$$

Example E. $\int x \ln x dx$. Let $u = \ln x$, $dv = x dx$ then $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$.

$$\int x \ln x dx = (\ln x) \frac{x^2}{2} - \int \frac{x^2}{2} \frac{dx}{x} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

Example F. $\int x \sin x dx$. Let $u = x$, $dv = \sin x dx$. Then $du = dx$ and $v = -\cos x$.

$$\int x \sin x dx = x(-\cos x) - \int -\cos x dx = -x \cos x + \sin x + c$$

Hint V: When choosing u and dv make sure dv is something that can be integrated. Also, the whole integrand should be taken up with u and dv .

Hint VI: Method should be used when integrand involves

(poly)trig	$e^{ax}(\sin bx) e^{ax} \cos bx$	$\sec^{(2n+1)} x$
(poly) $\ln x$	poly(inverse trig fact)	$\csc^{(2n+1)} x$

This is not an exclusive list!

Problems:

10. $\int \ln x dx$. Let $u = \ln x$, $dv = dx$.

11. $\int (x^2 + 2x - 1) \cos 3x dx$. Let $u = x^2 + 2x - 1$, $dv = \cos 3x dx$.
(Need to use integration by parts twice.)

12. $\int (x + 3)e^{2x} dx$. Let $u = x + 3$, $dv = e^{2x} dx$.

13. $\int x^2 \ln x dx$

14. $\int \tan^{-1} x dx$

15. $\int e^x \sin x dx$

Method 4. *Trigonometric integrals.* Integrands only involve trigonometric functions (not inverse trig functions!). Remember that certain trig functions “go together.”

$$\sin \theta \text{ and } \cos \theta$$

$$\tan \theta \text{ and } \sec \theta$$

$$\cot \theta \text{ and } \csc \theta$$

If you have mixed trig functions convert everything to $\sin \theta$ and $\cos \theta$. Another helpful identity is $\sin^2 \theta + \cos^2 \theta = 1$. From here you can derive $\tan^2 \theta + 1 = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$. You should also have the half angle formulas in your repertoire.

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

Most trigonometric integrations involve substitution. The only exceptions are when the integrand is $\sec^{2n+1} \theta$ or $\csc^{2n+1} \theta$ (see integration by parts) or $\sin^{2n} \theta \cos^{2m} \theta$ which involves the half angle formula (sometimes repeated several times).

Example G. $\int \sin \theta \cos^2 \theta d\theta$. Let $w = \cos \theta$; $dw = -\sin \theta d\theta$

$$\int \sin \theta \cos^2 \theta d\theta = \int -w^2 dw = -\frac{w^3}{3} + c = -\frac{\cos^3 \theta}{3} + c$$

Example H. $\int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + c$

Example I. $\int \sin^2 \theta \cos^3 \theta d\theta$ Let $w = \sin \theta$ then $dw = \cos \theta d\theta$.

$$\int \sin^2 \theta \cos^2 \theta \underbrace{\sin \theta d\theta}_{dw} = \int w^2 (1 - \sin^2 \theta) dw =$$

$$\int w^2 (1 - w^2) dw = \frac{w^3}{3} - \frac{w^5}{5} + c = \frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} + c$$

Example J. $\int \sin^2 \theta d\theta = \int \frac{1}{2}(1 - \cos 2\theta) d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c$

Hint VII: As you can see there's not a whole lot of guessing to do with substitution. For instance in Example I above if you had let $w = \cos \theta$ then $dw = -\sin \theta d\theta$ and you have an "extra" $\sin \theta$ in the integrand. Of course it would be silly to guess $w = \tan \theta$ or $\sec \theta$ or another trig function. Use common sense and a paper and pencil!

Problems:

16. $\int \tan \theta \sec^3 \theta d\theta$

19. $\int \sin \theta \cos \theta d\theta$

17. $\int \cos^2 \theta d\theta$

20. $\int \tan^3 2\theta \sec^2 2\theta d\theta$

18. $\int \cot \theta \csc^3 \theta d\theta$

21. $\int \sin^3 \theta \cos^4 \theta d\theta$

Note that integrands involving different arguments were not covered. i.e., $\int \cos 2x \sin 3x dx$. These type of integrals can be done using integration by parts twice and bringing the integral to the other side.

Method 5. *Trigonometric substitution.* You let $x =$ a trig function and change the integrand to one involving trig functions. The integrands involve no exponentials or trig functions to begin with. Rather they usually involve

$$\begin{aligned}(a^2 - x^2)^{n/2} \\ (x^2 - a^2)^{n/2} \\ (x^2 + a^2)^{n/2}\end{aligned}$$

where a is constant and n is an integer.

If the integrand involves:

The substitution is:

$$(a^2 - x^2)^{n/2}$$

$$x = a \sin \theta$$

$$(x^2 - a^2)^{n/2}$$

$$x = a \sec \theta$$

$$(a^2 + x^2)^{n/2}$$

$$x = a \tan \theta$$

Example K. $\int \frac{dx}{4+x^2}$. Let $x = 2 \tan \theta$ $dx = 2 \sec^2 \theta d\theta$.

$$\int \frac{dx}{4+x^2} = \int \frac{2 \sec^2 \theta d\theta}{4+4 \tan^2 \theta} = \int \frac{2 \sec^2 \theta d\theta}{4 \sec^2 \theta} = \frac{1}{2} \int d\theta = \frac{1}{2} \theta + c = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

Example L. $\int (1-4x^2)^{1/2} dx$ $2x = \sin \theta$ $2dx = \cos \theta d\theta$

$$\begin{aligned}\int (1-4x^2)^{1/2} dx &= \int (1-\sin^2 \theta)^{1/2} \frac{\cos \theta d\theta}{2} = \frac{1}{2} \int \cos^2 \theta d\theta = \frac{1}{2} \int \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{4} \theta + \frac{1}{8} \sin 2\theta + c \\ &= \frac{1}{4} \theta + \frac{1}{4} \sin \theta \cos \theta + c = \frac{1}{4} \sin^{-1}(2x) + \frac{1}{4} (2x)(1-4x^2)^{1/2} + c\end{aligned}$$

Problems:

$$22. \int \frac{dx}{(x^2-1)^{1/2}}$$

$$25. \int x^3(4+x^2)^{1/2} dx$$

$$23. \int \frac{dx}{x^2+2x+2}$$

$$26. \int \frac{dx}{\sqrt{2x-x^2}}$$

$$24. \int \frac{e^x dx}{1+e^{2x}}$$

Method 6. Partial Fractions. For this method the integrand can be written in the form $\frac{\text{polynomial}}{\text{polynomial}}$.

Steps:

1. Is the degree of the polynomial in the denominator greater than the degree of the polynomial in the numerator? If not, divide out.
2. Factor denominator.
3. Write the fraction as a sum of fractions.
4. Solve for constants.
5. Integrate.

Example M. $\int \frac{x-4}{x^2+5x+6} dx$

1. Yes.

2. $x^2 + 5x + 6 = (x + 2)(x + 3)$

3. $\frac{x-4}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$

4. $x - 4 = A(x + 3) + B(x + 2)$

$$\text{Let } x = -3 \quad -3 - 4 = B(-3 + 2) \Rightarrow B = 7$$

$$\text{Let } x = -2 \quad -2 - 4 = A(-2 + 3) \Rightarrow A = -6$$

5. $\int \frac{(x-4)}{x^2+5x+6} dx = \int \left(\frac{-6}{x+2} + \frac{7}{x+3} \right) dx = -6 \ln|x+2| + 7 \ln|x+3| + c$

Example N. $\int \frac{(x^2+2x-4)}{x^3+x} dx$

Steps:

1. Yes.

2. $x^3 + x = (x + 1)(x^2 - x + 1)$

3. $\frac{x^2+2x-4}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

4. $x^2 + 2x - 4 = A(x^2 - x + 1) + (Bx + C)(x + 1)$ Let $x = -1$ $1 - 2 - 4 = A(3) \Rightarrow A = \frac{-5}{3}$.

Set coefficients equal

$$\begin{array}{rcl} \text{coef of } x^2 & 1 & = A + B \\ \text{coef of } x & 2 & = -A + B + C \\ \text{const} & -4 & = A + C \end{array}$$

Using $A = -\frac{5}{3}$, $C = -\frac{7}{3}$, $B = \frac{8}{3}$.

$$5. \int \frac{x^2 + 2x - 4}{x^3 + x} dx = \int \frac{-\frac{5}{3}}{x+1} dx + \frac{1}{3} \int \frac{8x - 7}{x^2 - x + 1} dx$$

to integrate the 2nd integral $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$ so that $\frac{1}{3} \int \frac{8x - 7}{\frac{1}{4} + \left(x - \frac{1}{2}\right)^2} dx$.

Let $w = x - \frac{1}{2}$ and use trig substitution.

Problems:

$$27. \int \frac{(x^2 - 4)dx}{x^3 + 3x^2 + 2x}$$

$$28. \int \frac{x + 3}{x^3 + x} dx$$

$$29. \int \frac{x - 3}{x^3 + 4x^2 + 4x} dx$$

$$30. \int \frac{1}{x^3 - x} dx$$

Hint VIII: After factoring, the degree of the highest factor is usually not greater than 2.

Hint IX: Do get caught up on the idea that the original integrand can be written as a $\frac{\text{polynomial}}{\text{polynomial}}$. Each of the following can be put into this form using substitution.

Problems:

31. $\int \frac{e^x dx}{e^{2x} + 3e^x + 2}$. Let $u = 3e^x$.

32. $\int \frac{\sin x dx}{\cos^3 x + \cos x}$

33. $\int \frac{dx}{x^{5/2} + 7x^{3/2} + 12x^{1/2}}$

The above list is all of the “typical” methods taught. After this, you should not be afraid to just take a pencil and paper and try different things. A Friday night problem $\int \sqrt{\tan x} dx$.

Methods of Integration – Solutions

1. $e^x - \tan x + c$

2. $\frac{x^3}{3} - 2x + 4 \ln|x| + c$

3. $-\cos x + c$

4. $\ln(1 + \sin x) + c$

5. $\frac{1}{2} \sin(2x^3 + x^2 + 4) + c$

6. $\frac{1}{4}(\ln x)^4 + c$

7. $e^{\tan x} + c$

8. $\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c$

9. $\frac{1}{2} \ln|x^2 + 2x - 5| + c$

10. $x \ln x - x + c$

11. $\frac{1}{3}(x^2 + 2x - 1) \sin 3x$
 $+ \frac{2}{9}(x+1) \cos 3x - \frac{2}{27} \sin 3x + c$

12. $\frac{1}{2}(x+3)e^{2x} - \frac{1}{4}e^{2x} + c$

13. $\frac{x^3}{3} \ln x - \frac{x^3}{9} + c$

14. $x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + c$

15. $\frac{1}{2}(e^x \sin x - e^x \cos x) + c$

16. $\frac{1}{3} \sec^3 \theta + c$

17. $\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta + c$

18. $-\frac{1}{3} \csc^3 \theta + c$

19. $\frac{1}{2} \sin^2 \theta + c$

20. $\frac{1}{8} \tan^4 2\theta + c$

21. $\frac{1}{5} \cos^5 \theta - \frac{1}{7} \cos^7 \theta + c$

22. $\ln|x + \sqrt{x^2 - 1}| + c$

23. $\tan^{-1}(x+1) + c$

24. $\tan^{-1}(e^x) + c$

25. $32 \left[\frac{1}{5} \left(\frac{\sqrt{x^2 + 4}}{2} \right)^5 - \frac{1}{3} \left(\frac{\sqrt{x^2 + 4}}{2} \right) \right] + c$

26. $\sin^{-1}(x-1) + c$

27. $-2 \ln|x| + 3 \ln|x+1| + c$

28. $3 \ln|x| - \frac{3}{2} \ln|x^2 + 1| + \tan^{-1} x + c$

29. $-\frac{3}{4} \ln|x| + \frac{3}{4} \ln|x+2| - \frac{5}{2}(x+2)^{-1} + c$

30. $-\ln|x| + \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + c$

31. $-\ln|e^x + 2| + \ln|e^x + 1| + c$

32. $-\ln|\cos x| + \tan^{-1}(\cos x) + c$

33. $\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x^{1/2}}{3} \right) - \tan^{-1} \left(\frac{x^{1/2}}{2} \right) + c$