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These are the basic notes from the first day of class.

Differential equations are important in many aspects of engineering, science, economics, and everyday life. Anytime a quantity is changing with respect to time or position a differential equation could be involved. For example,

- how the population of fruit flies in your kitchen is increasing
- your speed as you came to school this morning
- how much interest your savings account is making
- how the strength of your coffee changes as the water filters down through the grounds.

DEFINITION. A *differential equation* (DE) is an equation involving an unknown function and its derivatives. If the unknown function depends on a single independent variable $(y(x))$ then only ordinary derivatives appear in the DE and we say it is an *ordinary differential equation* (ODE). If the unknown function depends on more than one independent variable $(u(x, y))$ then partial derivatives will appear and we say it is a *partial differential equation* (PDE).

EXAMPLES.

A. mass spring $mx'' + cx' + kx = 0$
$x\ldots$ is the position of the object from equilibrium at any time t
$m\ldots$ is the mass of the object
$c\ldots$ is the damping or viscosity
$k\ldots$ is the spring constant

B. population $dP / dt = a(1 - bP)P$
$P\ldots$ population at any time t
$a,b\ldots$ are positive constants and represent birth and death rates respectively

Notice that if $P = 1/b$ then $dP / dt = 0$ and the population is not changing. If $1 - bP < 0$ then dP / dt is negative and P is decreasing. If $1 - bP > 0$ then dP / dt is positive and P is increasing. These are examples of ODEs. Some examples of PDEs:

C. potential $u_{xx} + u_{yy} = 0$
$u\ldots$ potential at any x and y

D. diffusion or heat $u_t = \alpha^2 u_{xx}$
$u\ldots$ heat in a 1-dimensional bar at any x and t
$\alpha\ldots$ material constant

If derivatives are denoted by primes $(')$, dots $(\dot{})$, or $\frac{d}{dx}$, the equation must be an ODE. If derivatives are denoted by subscripts $(\ _x)$ or $\frac{\partial}{\partial y}$ the equation must be a PDE.

REMARK. $y^{(n)} = d^n y / dx^n$

State whether the following are ODE or PDEs also determine the independent variable and the dependent variable.

1. $x''' + 3tx'' + x = 0$

2. $x'' + x^4 = 0$

3. $x^{(4)} + x = 0$

4. $m\dfrac{d^2x}{dt^2} + t\sin t = 0$

5. $u_{yy} + u_{xx} + u_{tt} = 0$

6. $\dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0$

7. $y' + xy = 0$

8. $u_t + uu_x = 0$

What we want to be able to do is given a DE we would like to be able to solve for the unknown function.

Definition. The *order* of a DE is the highest number of derivatives that appear in any one term in the equation.

In the above examples we see that (A) is 2nd order, (B) is 1st order, (C) is 2nd order, and (D) is 2nd order. Determine the order in problems 1-8.

We will usually write $y^{(n)} = f(x, y, y', \ldots y^{(n-1)})$ where *f* is some known function of it arguments.

Definition. A *solution* of the ODE

$$y^{(n)} = f\left(x, y, y', \ldots y^{(n-1)}\right)$$

of the interval $\alpha < x < \beta$ is the function φ such that $\varphi', \varphi'', \ldots \varphi^{(n)}$ exist and satisfy

$$\varphi^{(n)} = f\left(x, \varphi, \varphi', \ldots, \varphi^{(n-1)}\right)$$

for all $x \in (\alpha, \beta)$.

Unless stated otherwise we assume that *f* is real-valued and we are interested only in real-valued solutions.

DEFINITION. The ODE

$$F\left(x, y, y', \ldots y^{(n)}\right) = 0$$

is said to be *linear* if *F* is linear in the variables $y, y', \ldots y^{(n)}$. If an equation is not linear we say it is *nonlinear*.

In problems (A)-(D), we have (A) is linear, (B) is nonlinear, and (C) and (D) are linear.

REMARK. If a *y* or derivatives of *y* appear in the equation, then the only coefficient of *y* or its derivatives can be functions of the independent variable..

EXAMPLES.

E. $y'' + x^2 y' = \sin x$ ODE, 2nd order, linear

F. $y^{(4)} + (\sin x) y = 0$ ODE, 4th order, linear

G. $y^{(5)} + \sin y = 0$ ODE, 5th order, nonlinear

H. $y'' + (y')^2 - y = 0$ ODE, 2nd order, nonlinear

I. $x^2 y''' + xy' + x^{1/2} y = 0$ ODE, 3rd order, linear

J. $u_{xx} + u_t = 0$ PDE, 2nd order, linear

K. $u_{xtx} + u_{xx} = 0$ PDE, 3rd order, linear

L. $u_{tt} + u_x u_{xxx} = 0$ PDE, 3rd order, nonlinear

Determine whether the problems in 1-8 are linear or nonlinear.