

A first order differential equation can be linear in either y or t . An ODE is linear in y if it can ultimately be written in the form

$$y' + p(t)y = f(t)$$

Steps: 1. Identify the coefficient of y' .

2. Is the derivative of this function equal to the coefficient of y ? If yes, go to 7. If no, go to 2.

3. Divide by coefficient of y' .

4. Now identify $p(t)$ (the coefficient of y).

5. Evaluate $\int_t p(s)ds$.

6. Multiply the ODE through by $e^{\int_t p(s)ds}$.

7. The left-hand side can be written as the product of the coefficient of y' times y . Write it like this.

8. Integrate both sides.

9. Plug in IC.

Example A. $t^2y' + 2ty = \sin t$. $y\left(\frac{\pi}{2}\right) = 1$.

1. The coefficient of y' is t^2 .

2. $\frac{d}{dt}t^2 = 2t = \text{coefficient of } y$. Go to 7.

7. $t^2y' + 2ty = \frac{d}{dt}(t^2y) = \sin t$

8. $\int d(t^2y) = \int \sin t dt$ so that $t^2y = -\cos t + c$.

9. Plug in IC: $\frac{\pi^2}{4} \cdot 1 = 0 + c$. Thus $t^2y = -\cos t + \frac{\pi^2}{4}$.

Example B. $(\sin t)y' + (\cos t)y = e^t \quad y\left(\frac{\pi}{4}\right) = 0$

1. The coefficient of y' is $\sin t$.
2. $\frac{d}{dt}\sin t = \cos t = \text{coefficient of } y$. Go to 7.
7. $(\sin t)y' + (\cos t)y = \frac{d}{dt}(y \sin t) = e^t$.
8. $\int d(y \sin t) = \int e^t dt$ so that $y \sin t = e^t + c$.
9. Plug in IC: $0 \sin \frac{\pi}{4} = e^{\pi/4} + c$. Thus $y \sin t = e^t - e^{\pi/4}$.

Example C. $2y' + y = t \quad y(0) = 1$.

1. The coefficient of y' is 2.
2. $\frac{d}{dt}2 = 0 \neq \text{coefficient of } y$.
3. Divide ODE by 2.
4. $p(t) = \frac{1}{2}$.
5. $\int \frac{1}{2} ds = \frac{1}{2}t$
6. $e^{t/2}y' + \frac{1}{2}e^{t/2}y = (e^{t/2})\frac{t}{2}$
7. $e^{t/2}y' + \frac{1}{2}e^{t/2}y = \frac{d}{dt}(e^{t/2}y) = (e^{t/2})\frac{t}{2}$
8. $\int d(e^{t/2}y) = \int \frac{t}{2}e^{t/2} dt$ so that

$$ye^{1/2t} = te^{1/2t} - 2e^{1/2t} + c$$

9. Plugging in IC $1 = 0 - 2 + c$. Thus $ye^{t/2} = te^{t/2} - 2e^{t/2} + 3$.

An equation is linear in t if it can be written as

$$t' + p(y)t = f(y).$$

The procedure is the same as the above replacing t by y and y by t .

Example D. $t' + (\tan y)t = \sin y$. $y(0) = 0$

1. The coefficient of t' is 1.
2. $\frac{d}{dy}(1) = 0 \neq \tan y = \text{coefficient of } t$.
3. Divide ODE by 1.
4. $p(y) = \tan y$
5. $\int_y p(s)ds = -\ln \cos y = \ln \sec y$
6. Note $e^{\ln \sec y} = \sec y$. This is what we will multiply the ODE through by
 $(\sec y)t' + (\tan y \sec y)t = \sin y \sec y$.
7. $(\sec y)t' + (\tan y \sec y)t = \frac{d}{dy}((\sec y)t) = \sin y \sec y$
8. $\int d(t \sec y) = \int \frac{\sin y}{\cos y} dy$ so that $t \sec y = -\ln \cos y + c$.
9. Plugging in IC: $0 \sec 0 = -\ln \cos 0 + c$. Thus $t \sec y = -\ln \cos y$.

Problems:

1. $e^{t^2} y' + 2te^{t^2} y = \tan t$ $y(0) = 1$
2. $t^{-3} y' - \frac{3}{t^4} y = t$ $y(1) = 3$
3. $(\sin y)t' + (\cos y)t = y^2 + 1$ $y(0) = \frac{\pi}{2}$
4. $y^2 t' + 2yt = \sin y$ $y(0) = \frac{\pi}{4}$
5. $(\cos t)y' + (\sin t)y = 1$ $y\left(\frac{\pi}{4}\right) = 2$
6. $t' + \frac{1}{y}t = \cos y$ $y(2) = 1$
7. $\frac{1}{2}y' - ty = e^{t^2} \sin^2 t$ $y(0) = 1$
8. $(\sin y)t' - (\cos y)t = \sin y$ $y(1) = \frac{\pi}{4}$

Hint I: Note that when an equation is linear in y , and you have written the left-hand side as the derivative of a product, the right-hand side should only involve t ! If not, something is wrong.

Hint II: By now you should realize that t , y , x , z , etc. are what we call “dummy” variables. You can have equations linear in any of these variables with one of the other variables the independent variable. A first order ODE is said to be linear if the ODE is linear in the dependent variable.

Examples:

$$9. \quad y^3 y' = \frac{y^4}{3t + y^5}$$

$$10. \quad z' + (\cot s)z = 1$$

$$11. \quad y^2 x' + 2yx = e^y$$

$$12. \quad t' = \frac{t}{t^3 - w}$$

$$13. \quad t' - \frac{1}{x}t = x^3 \sin x$$

Solutions:

$$1. \quad y = e^{-t^2} \ln |\sec t| + e^{-t^2}$$

$$2. \quad y = \frac{1}{2}t^5 + \frac{5}{2}t^3$$

$$3. \quad t = \frac{1}{3}y^3 \csc y + y \csc y - \left(\frac{\pi}{2} + \frac{1}{3} \left(\frac{\pi}{2} \right)^3 \right) \csc y$$

$$4. \quad t = -\frac{1}{y^2} \cos y + \frac{\sqrt{2}}{2y^2}$$

$$5. \quad y = \sin t + (2\sqrt{2} - 1) \cos t$$

$$6. \quad t = \sin y + \frac{1}{y} \cos y + (2 - \sin | - \cos |) \frac{1}{y}$$

$$7. \quad y = -2e^{t^2} \cos t + 3e^{t^2}$$

$$8. \quad t = -\sin y \ln |\csc y + \cot y| + (\sqrt{2} + \ln(\sqrt{2} + 1)) \sin y$$

$$9. \quad t = \frac{1}{2}y^5 + cy^3$$

$$10. \quad z = -\cot s + c \csc s$$

$$11. \quad x = y^{-2}e^y + cy^{-2}$$

$$12. \quad w = \frac{t^3}{4} + \frac{c}{t}$$

$$13. \quad t = -x^3 \cos x + 2x^2 \sin x + 2x \cos x + cx$$