Math 305 **Exact Equations**

A first order differential equation, written in the form

$$M(x,y) + N(x,y)y' = 0$$

is said to be exact if it can be written as the derivative of an implicit function. To determine if an equation is exact, and the process involved, is as follows

Steps:

1. Identify M and N.
2. Is $M_y = N_x$? If yes, continue.
3. There is a function ϕ such that
$$\phi_x = M$$
$$\phi_y = N.$$
4. Find the antiderivative of ϕ_x. $\phi = \int M dx + g(y)$.
5. Differentiate this expression with respect to y and set it equal to N.
6. You should now have $g'(y)$ be equal to a function of y only. Integrate to solve for g.
7. Solve for IC.

Example A. $2xy + \cos x + (x^2 + e^y)y' = 0 \qquad y(0) = 0$

1. $M = 2xy + \cos x, \qquad N = x^2 + e^y.$
2. $M_y = 2x, \qquad N_x = 2x$. They are equal.
3. $$\phi_x = 2xy + \cos x \ (= M)$$ $$\phi_y = x^2 + e^y \ (= N)$$
4. Find the antiderivative of ϕ_x to get
$$\phi = x^2y + \sin x + g(y)$$
5. Differentiate this with respect to y and set equal to N.
$$\phi_y = x^2 + g'(y) = x^2 + e^y$$
$$g'(y) = e^y$$
6. $g(y) = e^y + c$
7. $0^2 \cdot 0 + \sin 0 + e^0 + c = 0 \Rightarrow c = -1$. Thus $x^2y + \sin x + e^y - 1 = 0$.

Example B. $-\dfrac{1}{1+x^2} - \dfrac{y}{x^2} + ye^{xy} + (xe^{xy} + \dfrac{1}{x} + 3y^2)y' = 0 \qquad y(1) = 0$

1. $M = ye^{xy} - \dfrac{1}{1+x^2} - \dfrac{y}{x^2}$, $N = xe^{xy} + \dfrac{1}{x} + 3y^2$

2. $M_y = e^{xy} + xye^{xy} - \dfrac{1}{x^2}$, $N_x = e^{xy} + xye^{xy} - \dfrac{1}{x^2}$. They are equal.

3.
$$\phi_x = ye^{xy} - \frac{1}{1+x^2} - \frac{y}{x^2} \ (= M)$$
$$\phi_y = xe^{xy} + \frac{1}{x} + 3y^2 \ (= N)$$

4. Find the antiderivative of ϕ_x to get $\phi = e^{xy} - \tan^{-1} x + \dfrac{y}{x} + g(y)$

5. Differentiate this with respect to y and set equal to N.
$$\phi_y = xe^{xy} + \frac{1}{x} + g'(y) = xe^{xy} + \frac{1}{x} + 3y^2 \qquad g'(y) = 3y^2$$

6. $g(y) = y^3$

7. $e^0 - \tan^{-1} 1 + 0 + 0^3 = c$. Thus $e^{xy} - \tan^{-1} x + \dfrac{y}{x} + y^3 = 1 - \dfrac{\pi}{4}$.

Hint I: After you've done the first integration and then differentiated again, $g'(y) =$ only function of y! If x's appear, you've made a mistake!

Hint II: In step 4, you could find the antiderivative of the ϕ_y equation. In this case, the "constant" is a function of x. You then differentiate this expression with respect to x.

Problems:

1. $y + 2xe^{x^2} + (x + \cos y)y' = 0 \qquad y(0) = \dfrac{\pi}{2}$

2. $\dfrac{x}{x^2+y^2} + \dfrac{yy'}{x^2+y^2} + 1 - y' = 0 \qquad y(1) = 1$

3. $ye^{xy} + 2x - 2xy + (xe^{xy} + 3y^2 - x^2)y' \qquad y(0) = 1$

4. $-\sin y y' + e^x + \dfrac{\cos\frac{x}{y}}{y} - \dfrac{x\cos\frac{x}{y}}{y^2}y' = 0 \qquad y(0) = \dfrac{\pi}{2}$

5. $(3x^2y^2 + 3y^2 + y)dx + (2x^3y + 6xy + x)dy = 0 \qquad y(1) = 2$

Solutions:

1. $xy + e^{x^2} + \sin y = 2$

2. $\dfrac{1}{2}\ln(x^2+y^2) + x - y = \ln\sqrt{2}$

3. $e^{xy} + x^2 - x^2y + y^3 = 2$

4. $e^x + \cos y + \sin\dfrac{x}{y} = 1$

5. $x^3y^2 + 3xy^2 + xy = 12$