Math 305 **Higher Order Nonlinear**

When trying to solve higher order nonlinear problems i.e., equations of the form $y'' = f(x, y, y')$ the level of difficulty increases quickly. We will be concerned with two very special cases. The first one is when no y's appear explicitly in the equation. The second one is when no x's appear explicitly in the equation. In both cases a change of variables is made to reduce the equation to a first order equation.

Case 1. $y'' = f(x, y')$

Let $y' = v(x)$. Then $\frac{d}{dx}y' = \frac{d}{dx}v \Rightarrow y'' = \frac{dv}{dx}$ and the DE becomes

$$\frac{dv}{dx} = f(x, v)$$

Example A. $y'' - \left(2x + \frac{1}{x}\right)y' = 0 \quad y(1) = 3 \quad y'(1) = 2$ No y's appear.

1. Let $y' = v(x)$.
2. $\frac{d}{dx}y' = y'' = \frac{dv}{dx}$.
3. Substitute for y'' and y'. Then $\frac{dv}{dx} - \left(2x + \frac{1}{x}\right)v = 0$.
4. Solve ODE. This equation is both linear and separable and can be solved either way. Using separation.

$$\frac{dv}{v} = \left(2x + \frac{1}{x}\right)dx$$
$$\ln v = x^2 + \ln x + c$$

5. Plug in IC. $x = 1,\ y = 3,\ v = 2$ and $\ln 2 = 1 + \ln 1 + c \Rightarrow c = \ln 2 - 1$.
6. Solve for v. $e^{\ln v} = v = e^{x^2 + \ln x + \ln 2 - 1} = 2xe^{x^2-1}$
7. Substitute. $v = y' = \frac{dy}{dx} = 2xe^{x^2-1}$
8. Solve the ODE. This is a separable equation so that $y = e^{x^2-1} + c_2$.
9. Plug in IC. $x = 1\ y = 3\ 3 = 1 + c_2$ and $y = e^{x^2-1} + 2$

Example B. $y'' + (y')^2 = 0 \quad y(1) = 3 \quad y'(1) = -1$

1. Let $y' = v(x)$.

2. Then $\frac{d}{dx}y' = y'' = \frac{dv}{dx}$.

3. Substitute for y'' and y'. The $\frac{dv}{dx} + v^2 = 0$.

4. Solve ODE. Solve this separable equation to get

$$\frac{1}{v} = x + c$$

5. Plug in I.C. $x = 1,\ y' = -1 \quad -1 = 1 + c \Rightarrow c = -2$.

6. Solve for v. $v = \frac{1}{x-2}$.

7. Substitute $\frac{dy}{dx} = v = \frac{1}{x-2}$.

8. Solve the ODE. This is a separable equation and $\ln|x-2| = y + c_2$.

9. Plug in IC. $\ln|1-2| = 3 + c_2 \Rightarrow c_2 = -3$ and $\ln|2-x| = y - 3$.

Problems:

1. $x^2 y' y'' + x(y')^2 = x\sin^4 x + 2x^2 \sin^3 x \cos x$ with $y(\pi) = 4 \quad y'(\pi) = 0$

2. $xy'' = y' + x\tan\frac{y'}{x}$ with $y(1) = 1,\ y'(1) = \frac{\pi}{2}$

3. $y'' = e^{x-y'}$ with $y(0) = 3 \quad y'(0) = 0$

4. $-xy'' + y' = e^x (y')^2$ with $y(1) = 2 \quad y'(1) = e^{-1}$

The second kind of substitution is used when $y,\ y',\ y''$ appear explicitly in the equation but x does not. In this case let $y' = v(y(x))$ then $\frac{d}{dx}y' = y'' = \frac{d}{dx}v(y(x)) = \frac{dv}{dy}\cdot\frac{dy}{dx} = \frac{dv}{dy}y' = \frac{dv}{dy}\cdot v$. (By Chain Rule)

<u>Note:</u> Before $y'(x) = v(x)$. Now $y'(x) = v(y(x))$. Thus $v\frac{dv}{dy} = f(y, v)$.

Example C. $y'' - 2yy' = 0 \quad y(0) = 1 \quad y'(0) = 2$

1. Let $y'(x) = v(y(x))$.
2. $\frac{d}{dx}y' = y'' = \frac{d}{dx}v(y(x)) = \frac{dv}{dy}\cdot\frac{dy}{dx} = y'\frac{dv}{dy} = v\frac{dv}{dy}$.
3. Substitute for y'' and y'. Then $v\frac{dv}{dy} - 2yv = 0$.
4. Solve ODE. This is a separable equable and $v(y) = y^2 + c$.
5. Plug in IC. $2 = 1 + c \Rightarrow c = 1$ and
6. Solve for v. $v = y^2 + 1$
7. Substitute $v = \frac{dy}{dx} = y^2 + 1$.
8. Solve ODE. This is also a separable equation and $\tan^{-1} y = x + c_2$.
9. Plug in IC. $\Rightarrow \tan^{-1} 1 = 0 + c_2 \Rightarrow c_2 = \frac{\pi}{4}$ Thus $\tan^{-1} y = x + \frac{\pi}{4}$.

Example D. $y''y^2 + 2y(y')^2 = 0 \quad y(0) = 1 \quad y'(0) = 1$

1. Let $y' = v(y(x))$.

2. $\frac{d}{dx}y' = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$.

3. Substitute for y'' and y'. Then $v\frac{dv}{dy}y^2 + 2yv^2 = 0$.

4. Solve ODE. This is a separable equation. $\frac{dv}{v} = \frac{-2}{y}$ and $\ln v = -2\ln y + c$.

5. Plug in IC. $\ln 1 = -2\ln 1 + c = 0$ so $\ln v = -2\ln y$.

6. Solve for v. $v = \frac{1}{y^2}$.

7. Substitute $v(y(x)) = y'$ and $\frac{dy}{dx} = \frac{d=1}{y^2}$.

8. Solve ODE. This is a separable equation and $\frac{y^3}{3} = x + c$.

9. Plug in IC. $\frac{1}{3} = 0 + c$ and $\frac{y^3}{3} = x + \frac{1}{3}$.

Problems:

5. $2yy'' + (y')^2 = 0$ with $y(2) = 1 \quad y'(2) = -1$

6. $yy'' + (y')^2 = 1$ with $y(0) = 1,\ y'(0) = \sqrt{2}$

7. $y'' - 2yy' = 0$ with $y(3) = \sqrt{5} \quad y'(3) = 1$