Math 305 **Second Order, Linear, Homogeneous, Constant Coefficients**

Second order, linear, constant coefficients, homogeneous equations can be written in the form

$$ay'' + by' + cy = 0.$$

Look for solutions of the form $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$ and the ODE becomes

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Divide by e^{rt}, the characteristic equation is

$$ar^2 + br + c = 0.$$

Case 1. $b^2 - 4ac > 0$. Let $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$. The two "different" solutions are e^{r_1t} and e^{r_2t} and the homogeneous equation is $y_n = c_1e^{r_1t} + c_2e^{r_2t}$.

Example A. $y'' + 5y' + 6y = 0$

1. Characteristic equation: $r^2 + 5r + 6 = 0$
2. Roots: $r = -2, -3$.
3. Different Solutions: e^{-2t}, e^{-3t}
4. Homogeneous equation: $y_n = c_1e^{-2t} + c_2e^{-3t}$

Example B. $y'' + 5y' - 21y = 0$

1. Characteristic equation: $r^2 + 4r - 21 = 0$
2. Roots: $r = -7, 3$.
3. Different Solutions: e^{-7t}, e^{3t}
4. Homogeneous equation: $y_n = c_1e^{-7t} + c_2e^{3t}$

Case 2. $b^2 - 4ac < 0$. Let $\alpha = -\dfrac{b}{2a}$ and $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$. The two "different" solutions are $e^{\alpha_t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$ and the homogeneous equation is $y_n = e^{\alpha t}[c_1 \cos \beta t + c_2 \sin \beta t]$.

Example C. $y'' + 9y = 0$

1. Characteristic equation: $r^2 + 9 = 0$
2. Roots: $r = \pm 3i \quad \alpha = 0,\ \beta = 3.$
3. Different Solutions: $\cos 3t \quad \sin 3t$
4. Homogeneous equation: $y_n = c_1 \cos 3t + c_2 \sin 3t$

Example D. $y'' + 2y' + 3y = 0$

1. Characteristic equation: $r^2 + 2r + 3 = 0$
2. Roots: $r = \dfrac{-2 \pm \sqrt{4 - 12}}{2} \quad \alpha = -1$ and $\beta = \sqrt{2}$.
3. Different Solutions: $e^{-t} \sin \sqrt{2}t$ and $e^{-t} \cos \sqrt{2}t$
4. Homogeneous equation: $y_n = e^{-t}[c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t]$

Case 3. $b^2 - 4ac = 0$ then $r = -\dfrac{b}{2a}$ The two "different" solutions are e^{rt} and te^{rt} and the homogeneous solution is $y_n = c_1 e^{rt} + c_2 te^{t}$.

Example E. $y'' - 8y' + 16y = 0$

1. Characteristic equation: $r^2 - 8r + 16 = 0$
2. Roots: $r = 4$.
3. Different Solutions: e^{4t} and te^{4t}
4. Homogeneous equation: $y_n = c_1 e^{4t} + c_2 te^{4t}$

Example F. $y'' - 2\pi y' + \pi^2 y = 0$

1. Characteristic equation: $r^2 - 2\pi r + \pi^2 = 0$
2. Roots: $r = \pi$.
3. Different Solutions: $e^{\pi t}$ and $te^{\pi t}$
4. Homogeneous equation: $y_n = c_1 e^{\pi t} + c_2 te^{\pi t}$

Problems: Find the homogeneous solution to the

1. $3y'' + 2y' + y = 0$

2. $y'' - y' - 20y = 0$

3. $y'' + 3y' = 0$

4. $2y'' + 2y' + y = 0$

5. $y'' + 16y = 0$

6. $5y'' + 4y' + y = 0$

7. $y'' + 4y' + 4y = 0$

8. $y'' - 10y' + 25y = 0$

9. $y'' + 2\sqrt{2}y' + 2y = 0$

10. $y'' + 7y' + 2y = 0$

11. $3y'' + 2y' - 8y = 0$

12. $y'' + 2ey' + e^2y = 0$

13. $3y'' - 2y' + 4y = 0$

14. $y'' + 6y' + 6y = 0$

15. $2y'' + 4y' + 7y = 0$