Math 305 **Variation of Parameters**

Variation of parameters can be used on any linear nonhomogeneous equations. We will restrict ourselves here to second order i.e.,

$$y'' + p(t)y' + q(t)y = f(t).$$

1. Solve the homogeneous equation $(y'' + p(t)y' + q(t)y = 0)$.

$$y_n = c_1 y_1 + c_2 y_2.$$

2. Make sure the coefficient of y'' is 1.

3. Look for the particular solution to be of the form

$$y_p = v_1 y_1 + v_2 y_2$$

   where y_1 and y_2 are from 1, above, and you need to determine v_1 and v_2.

4. Solve

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= f(t). \end{aligned}$$

5. The general solution is $y_g = y_n + y_p$.

Example A. $y'' + y = \csc 2t$.

1. Solve $y'' + y = 0$. The characteristic equation is $r^2 + 1 = 0$ so $r = \pm i$

$$y_n = c_1 \cos t + c_2 \sin t.$$

2. The coefficient of y'' is 1.

3. $y_p = v_1 \cos t + v_2 \sin t$.

4. Need to solve

$$\begin{aligned} v_1' \cos t + v_2' \sin t &= 0 \\ v_1'(-\sin t) + v_2' \cos t &= \csc 2t. \end{aligned}$$

   Multiply equation 1 by $\sin t$, equation 2 by $\cos t$ and add the two equations to eliminate v_1'

$$\begin{aligned} v_2' = \csc 2t \cos t = \frac{\cos t}{csc2t} &= \frac{\cos t}{2 \sin t \cos t} \\ &= \frac{1}{2} \csc t. \end{aligned}$$

   Multiply equation 1 by $\cos t$, equation 2 by $(-\sin t)$ and add the two equations to eliminate v_2'

$$v_1' = \frac{-\sin t}{csc2t} = -\frac{1}{2} \sec t.$$

   Integrate v_1' and v_2' to get $v_1 = -\frac{1}{2} \ln|\sec t + \tan t|, v_2 = -\frac{1}{2} \ln|\csc t + \cot t|$.

5. $y_g = c_1 \cos t + c_2 \sin t - \frac{1}{2} \ln|\sec t + \tan t| \cos t + \frac{1}{2} \csc t \sin t$

Example B. $y'' - 2y' + y = e^t \ln t$.

1. Solve $y'' - 2y' + y = 0$. The characteristic equation is $r^2 - 2r + 1 = 0$ so $r = 1$ is a double root and

$$y_n = c_1 e^t + c_2 t e^t.$$

2. The coefficient of y'' is 1.

3. $y_p = v_1 e^t + v_2 t e^t$.

4. Need to solve

$$v_1' e^t + v_2' e^t t = 0$$
$$v_1' e^t + v_2'(e^t + te^t) = e^t \ln t.$$

   Subtract equation 1 from equation 2 to see

$$v_2' e^t = e^t \ln t \text{ so}$$
$$v_2' = \ln t.$$

   From equation 1 $v_1' = -tv_2' = -t \ln t$.
   Thus $v_1' = -\frac{t^2}{2} \ln t + \frac{t^2}{4}$ and $v_2 = t \ln t - t$.

5. $y_g = c_1 e^t + c_2 t e^t + \left(-\frac{t^2}{2} \ln t + \frac{t^2}{4}\right) e^t + (t \ln t - t) t e^t$

Problems:

1. $y'' - 2y' + y = e^t \tan t \sec t$

2. $x^2 y'' + xy' - y = \frac{x^2}{x^2 + x^3}$ $\rightarrow$ (one solution to the homogeneous equation for both is $y_1 = x$).

3. $x^2 y'' - 2xy' + 2y = x \ln x$