Math 305 **Fourier Series**

If f is periodic of period 2L and f and f' are piecewise continuous, then $f(x)$ can be written as a Fourier Series. i.e.

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where $a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$ and $b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$.

Example A. $f(x) = \begin{cases} 1 & -3 \leq x < 0 \\ 0 & 0 \leq x < 3 \end{cases}$ $\qquad f(x+6) = f(x)$. Here $L = 3$

$$a_n = \frac{1}{3} \int_{-3}^{3} f(x) \cos \frac{n\pi x}{3} dx = \frac{1}{3} \left[\int_{-3}^{0} 1 \cos \frac{n\pi x}{3} dx + \int_{0}^{3} 0 \cos \frac{n\pi x}{3} ds \right]$$

If $n = 0,\ a_0 = \dfrac{1}{3} \displaystyle\int_{-3}^{0} 1 dx = \frac{1}{3}(0 - (-3)) = 1$

$$n \neq 0,\ a_n = \frac{1}{3} \int_{-3}^{0} \cos \frac{n\pi x}{3} dx = \frac{1}{3} \left(\frac{3}{n\pi} \right) \sin \frac{n\pi x}{3} \bigg|_{-3}^{0} = \frac{1}{n\pi} [\sin 0 - \sin(-n\pi)] = 0$$

$$b_n = \frac{1}{3} \int_{-3}^{3} f(x) \sin \frac{n\pi x}{3} dx = \frac{1}{3} \left[\int_{-3}^{0} 1 \sin \frac{n\pi x}{3} dx + \int_{0}^{3} 0 \sin \frac{n\pi x}{3} dx \right]$$

$$= \frac{1}{3} \left(\frac{-3}{n\pi} \right) \cos \frac{n\pi x}{3} \bigg|_{-3}^{0} = -\frac{1}{n\pi} [\cos 0 - \cos(-n\pi)] = -\frac{1}{n\pi} [1 - \cos n\pi]$$

$$f(x) = \frac{1}{2} + \sum_{1}^{\infty} -\frac{1}{n\pi} [1 - \cos n\pi] \sin \frac{n\pi x}{3}$$

Example B. $f(x) = x^2 \quad -2 \le x \le 2, \qquad f(x+4) = f(x)$. Here $L = 2$.

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-2}^{2} x^2\cos\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left[x^2\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{2} - \int_{-2}^{2} 2x\frac{2}{n\pi}\sin\frac{n\pi x}{2}dx\right]$$

$$= \frac{1}{2}\left[0 - \frac{4}{n\pi}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\Big|_{-2}^{2} + \int_{-2}^{2}\frac{2}{n\pi}\cos\frac{n\pi x}{2}dx\right]\right]$$

$$= -\frac{2}{n\pi}\left[-\frac{2}{n\pi}(2\cos n\pi - (-2)\cos(-n\pi)) + \left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi x}{2}\Big|_{-2}^{2}\right]$$

$$= +\left(\frac{2}{(n\pi)}\right)^2[4\cos n\pi]$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} x^2 dx = \frac{8}{3}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-2}^{2} x^2\sin\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left[x^2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\Big|_{-2}^{2} + \int_{-2}^{2} 2x\left(+\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)dx\right]$$

$$\frac{1}{2}\left[0 + \frac{4}{n\pi}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)\Big|_{-2}^{2}\frac{2}{n\pi}\sin\frac{n\pi x}{2}dx\right]\right]$$

$$= \frac{1}{2}\left[0 + \frac{16}{(n\pi)^2}\cos\frac{n\pi x}{2}\Big|_{-2}^{2}\right] = 0$$

$$f(x) = \frac{\frac{8}{3}}{2} + \sum_1^\infty\left(\frac{4}{(n\pi)^2}4\cos n\pi\right)\cos\frac{n\pi x}{2}$$

Problems:

1. $f(x) = \begin{cases} 1 & -2 \le x < 0 \\ 2 & 0 \le x < 2 \end{cases} \qquad f(x+4) = f(x)$

2. $f(x) = -x \quad -3 \le x < 3 \quad f(x+6) = f(x)$

3. $f(x) = \begin{cases} -x & -1 \le x < 0 \\ x & 0 \le x < 1 \end{cases} \qquad f(x+2) = f(x)$

If f is an odd function then $f(-x) = -f(x)$ and there are no a_n terms. In addition,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

If f is an even function then $f(-x) = -f(x)$ and there are no b_n terms. In addition,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

In the above example where $f(x) = x^2$, it is not surprising that $b_n = 0$.

It is sometimes necessary to take a function defined on an interval $[0, L]$ and extend it as an odd or even function around the origin and then make the function periodic.

For example,
$f(x) = \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x < 2 \end{cases}.$

If f is extended as an odd function around 0, then it would look like

$$f(x) = \begin{cases} -2 - x & -2 \le x < -1 \\ x & -1 \le x < 1 \\ 2 - x & 1 \le x < 2 \end{cases} \qquad \text{and } f(x+4) = f(x)$$

.

You can now find the Fourier Series of this function.

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx$$

$$\frac{1}{2}\left[\int_{-2}^{-1}(-2-x)\sin\frac{n\pi x}{2}dx + \int_{-1}^{1} x\sin\frac{n\pi x}{2}dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}dx\right]$$

$$= \frac{2}{2}\left[\int_{0}^{1} x\sin\frac{n\pi x}{2}dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}dx\right]$$

If f is extended as an even function around 0, then it would look like

$$f(x) = \begin{cases} +2+x & -2 \le x < 1 \\ -x & -1 \le x < 0 \\ x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \end{cases} \qquad f(x+4) = f(x).$$

The Fourier coefficients are

$$a_n = \frac{1}{2}\left[\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{-1}(2+x)\cos\frac{n\pi x}{2}dx + \int_{-1}^{0} -x\cos\frac{n\pi x}{2}dx + \int_{0}^{1} x\cos\frac{n\pi x}{2}dx = \int_{1}^{2}(2-x)\cos\frac{n\pi x}{2}dx\right]$$

$$= \frac{2}{2}\left[\int_{0}^{1} x\cos\frac{n\pi x}{2}dx + \int_{1}^{2}(2-x)\cos\frac{n\pi x}{2}dx\right]$$

Problems:

4. Extend f oddly $f(x) = x(x-1)\ \ 0 \le x < 1$.

5. Extend f evenly $f(x) = x^2 - 4\ \ 0 \le x < 2$.