On the natural endomorphism field on a Lie algebra

Jerzy Kocik

Department of Mathematics, SIU, Carbondale, IL62901
e-mail: jkocik@siu.edu

Abstract: An 'endomorphism field' on a Lie algebra is introduced. It is defined as the $(1,1)$-type tensor field naturally determined by the Lie structure, satisfying a certain Nijenhuis bracket condition.

The Lie endomorphism field has connections with dynamical systems. Here we show its relevance for Lax equations and show that the space of Lax vector fields is closed under operation of Lie bracket and introduce an analog of Poisson bracket for vector fields on a Lie algebra. The bracket turnes the space of vector fields into a Lie algebra.

(These results may be considered as a construction complementary to the Kirillov-Kostant-Souriau theorem on symplectics geometry of coadjoint orbits.)

Key words: Lie algebra, Nijenhuis-Schouten bracket, Lax equations, Kirillov-Kostant-Souriau theorem

Notation. We shall distinguish between purely algebraic and a differential products by using two types of brackets;

$[\![\ ,\ ]\!]$ — Lie algebra product.

$[\ ,\ ]$ — commutator of vector fields or tensor fields (Schouten bracket, Nijenhuis bracket).

1. Introduction

It is well-known [Tu, Li] that the underlying space L^* of a dual Lie algebra L possesses a natural Poisson structure in terms of a smooth bi-vector field $\Omega \in \wedge^2 TL^*$, which satisfies the Jacobi condition $[\Omega, \Omega] = 0$, and, when restricted to coadjoint orbits, is nondegenerate and therefore invertible into a symplectic structure. Existence of these natural symplectic sheets is the content of the Kirillov-Kostant-Souriau Theorem (see, e.g., [Ki], [So], [CIMP]).

It seems overlooked that the Lie algebra L itself possesses a natural differential-geometric object, namely a $(1,1)$-type tensor field that we shall call a Lie endomorphism field, $\mathcal{A} \in \mathcal{T}^{(1,1)}L$. This paper is to call attention to this object. In particular, we show its properties and connections with other topics like Lax equations, Lagrange mechanics, and the generalized Euler top. The major property of $\mathcal{A}$ is that it is proportional to its Nijenhuis derivative; this allows one to introduce a new bracket of vector fields, which is for Lax equations — what the Poisson bracket is for Hamiltonian vector fields.

The range of problems that arise is wider and includes representation theory, quantization schemes, etc., these however will be discussed elsewhere.

2. The natural endomorphism field on a Lie algebra

Let L be a Lie algebra with the product $L \times L \mapsto L$ denoted $[\![v, w]\!]$ (double bracket to discriminate it from the commutator of vector fields, denoted simply $[\ ,\ ]$). Lie algebra bracket is bilinear, skew-symmetric (i), and satisfies the Jacobi identity (ii):

$$\begin{array}{rl} (i) & [\![v, w]\!] = -[\![w, v]\!] \\ (ii) & [\![v, [\![w, z]\!]]\!] + [\![w, [\![z, x]\!]]\!] + [\![x, [\![w, z]\!]]\!] = 0 \end{array} \tag{2.1}$$

In a basis $\{e_i\}$, the commutator can be represented via “structure constants”:

$$[\![e_i, e_j]\!] = c_{ij}^k e_k \tag{2.2}$$

The space L underlying the Lie algebra can be treated as a flat manifold admitting differential-geometric objects. Denote by μ_x the natural isomorphism of linear spaces $\mu_x : T_xL \to L$ for any $x \in L$.

Theorem 2.1 *The manifold of the Lie algebra L possesses a natural field of endomorphism $\mathcal{A} \in \mathcal{T}_1^1 L$ defined*

$$\mathcal{A}(v) = \mu_x^{-1} \circ \mathrm{ad}_x \circ \mu_x(v) \tag{2.3}$$

The endomorphism field $\mathcal{A}$ is restrictable to adjoint orbits on L, and its Nijenhuis derivative $[\mathcal{A},\mathcal{A}]$ is a vector-valued biform

$$\begin{aligned}[\mathcal{A},\mathcal{A}](v,w) &= -2\mathcal{A}([\![v,w]\!]) \\ &= -2([\![\mathcal{A}v,w]\!]+[\![v,\mathcal{A}w]\!])\end{aligned} \tag{2.4}$$

for any $v,w\in TL$.

(Proof is given in the next section). In the coordinate description, the endomorphism field is a a (1,1)-variant tensor field:

$$\mathcal{A} = x^i\, c_{ij}^k\; dx^j\otimes\partial_k. \tag{2.5}$$

and can be described as local transformations that at point $x\in L$ can be represented by matrix $\mathcal{A}_j^k(x) = x^i\, c_{ij}^k$. The Nijenhuis bracket (2.4) equals

$$[\mathcal{A},\mathcal{A}] = -x^k\; c_{kp}^i c_{ab}^p\; (dx^a\wedge dx^b)\otimes\partial_i \tag{2.6}$$

In the following we shall prove the theorem in the the context of geometry of Lie algebra.

3. The geometry of a Lie algebra

Let $\{e_i\}$ be a basis in L and $\{x^i\}$ be the associated coordinate system on L. Define a (constant) $(1,2)$-type tensor field on the manifold L by

$$\lambda = \tfrac{1}{2}c_{ij}^k\; dx^i\wedge dx^j\otimes\partial_k \tag{3.1}$$

where c_{ij}^k are the structure constants (2.1) (for more on this approach see [Ko]). As a linear space, manifold L is equipped with a natural vector field, the Liouville vector field, which in a linear coordinate system is

$$J = x^i\partial_i \tag{3.2}$$

The endomorphism field (2.3) may be defined as

$$\mathcal{A} = J\,\lrcorner\,\lambda \tag{3.3}$$

Denote $\tilde{v}$ a constant vector field on L that at the origin takes value $v\in L$ (in coordinates, $\tilde{v} = v^i\partial_i$, if $v = v^ie^i$). Endomorphism field $\mathcal{A}$ determines a representation of Lie algebra L in terms of vector fields on L: with every algebra element $v \in L$, associate a vector field

$$\begin{aligned} X_v &= \mathcal{A}\tilde{v} \\ &= x^i v^j c_{ij}^k \partial_k \end{aligned} \tag{3.4}$$

The map $v \to X_v$ forms the *infinitesimal representation* of L in $\mathcal{X}L$

$$[X_v, X_w] = X_{[\![v,w]\!]} \tag{3.5}$$

and the homomorphism readily follows from the Jacobi identity.

Corollary 3.1 The following are convenient formulae

$$\begin{aligned} (i) &\quad [X_v,\, X_w] = X_{[\![v,w]\!]} \\ (ii) &\quad [X_v,\, \widetilde{w}] = \widetilde{[\![v,w]\!]} \\ (iii) &\quad [\![\widetilde{v},\widetilde{w}]\!] = \widetilde{[\![v,w]\!]} \\ (iv) &\quad [\widetilde{v},\widetilde{w}] = 0 \end{aligned}$$

The image of $\mathcal{A}$ spans at every point a subspace $\mathcal{D} \subset TL$ of the tangent space of L. The integral manifolds of this distribution coincide with the adjoint orbits determined by action of a Lie group on Lie algebra. Here, however, we define "adjoint orbits" without reference to the Lie group, just as the integral manifolds of $\mathcal{D}$. Thus, at any point $x \in \mathcal{O} \subset L$,

$$T\mathcal{O} = \operatorname{Im} \mathcal{A} = \operatorname{span}\{X_v \mid v \in L\} \tag{3.6}$$

Now we are ready to sketch the proof of the theorem.

Proof of Theorem 2.1: First, let us calculate the Nijenhuis bracket. Recall that that Nijenhuis bracket $[K,K]$ of a vector-valued one-form (endomorphism field) with itself is a vector-valued bi-form that, evaluated on two fields X and Y, takes the value

$$\frac{1}{2}[K,K](X,Y) = [KX,KY] - K[KX,Y] - K[X,KY] + K^2[X,Y] \tag{3.7}$$

(see, e.g., [Mi]). To get result (3.4), evaluate (half of) thee Nijenhuis bracket $[\mathcal{A},\, \mathcal{A}]$ on two constant vector fields $\tilde{v}$ and $\tilde{w}$. Using formulae of Corollary 3.1, one gets

$$\begin{aligned} \frac{1}{2}[\mathcal{A},\mathcal{A}](\tilde{v},\tilde{w}) &= [X_v, X_w] - \mathcal{A}([X_v,\tilde{w}]) - \mathcal{A}([\tilde{v},X_w]) + 0 \\ &= X_{[\![v,w]\!]} - X_{[\![v,w]\!]} - X_{[\![v,w]\!]} \\ &= -X_{[\![v,w]\!]} \end{aligned}$$

In particular, substitution $X = \partial_a$ and $Y = \partial_b$ leads to the coordinate formula (2.6). Now, let us show that $\mathcal{A}$ can be restricted to orbits, i.e.,

$$\mathcal{A}(T_x\mathcal{O}) \subset T_x\mathcal{O}$$

for each point $x \in \mathcal{O}$. Rewrite (3.3) for $X \in T_xL$

$$\mathcal{A}(X) = \mu_x^{-1}\left(\llbracket x, (\mu_x(X)\rrbracket\right)$$

Vector of the vector field X_v at point $x \in L$ can be expressed as

$$X_v(x) = \mathcal{A}(v) = \mu_x^{-1}\left(\llbracket x, v\rrbracket\right)$$

Now

$$\begin{aligned}\mathcal{A}(X_v) &= \mu_x^{-1}\left(\llbracket x, \mu_x(X_v)\rrbracket\right)\\ &= \mu_x^{-1}\left(\llbracket x, \llbracket x, v\rrbracket\rrbracket\right)\\ &= X_{\llbracket x,v\rrbracket} \in T_x\mathcal{O}\end{aligned}$$

which was to be proven. □

Example 1: Consider the 2-dimensional nilpotent algebra defined by $[e_1, e_2] = e_2$. Then

$$\mathcal{A} = x_1\; dx_2 \otimes \partial_2$$
$$[\mathcal{A}, \mathcal{A}] = -2x_1\;(dx_1 \wedge dx_2) \otimes \partial_2$$

Example 2: The Lie algebra of 3-dimensional rotations, so_3, is defined by relations $[e_i, e_j] = \varepsilon_{ijk}\, e_k$. Thus

$$\mathcal{A} = x_1\;(dx_2 \otimes \partial_3 - dx_3 \otimes \partial_2) + (\text{cyclic terms})$$
$$[\mathcal{A}, \mathcal{A}] = dx_1 \wedge dx_2 \otimes (x_1\partial_2 - x_2\partial_1) + (\text{cyclic terms})$$

On a unit sphere defined by the Killing form, tensor $\mathcal{A}$ forms an almost complex structure, $\mathcal{A} \circ \mathcal{A} = -\mathrm{id}$.

Example 3: Let L be a 3-dimensional linear space endowed with a (pseudo-) Euclidean scalar product g. Define a bracket in a pseudo-Euclidean space by:

$$[v, w] = \star(v \wedge w) \tag{3.8}$$

where the Hodge star is determined by defining its value on an i-vector as an $(3-i)$-vector satisfying

$$\eta\;(\star v \wedge w) = g(v, w) \qquad \forall w \in \wedge^i L$$

Formula (3.8) defines a Lie structure in L; it can also be expressed by a tensor

$$\lambda = *\mathrm{id}$$

with the Hodge star is acting here on the covariant part of $\mathrm{id} = e_i \otimes \varepsilon^i$ only. (Another form of (7) is $\lambda = g^{ij}\; e_i \otimes e_j \lrcorner \eta$). The Killing form is $\kappa = -g$. Particular cases of the signature of g correspond to Lie algebras $o(3)$, $o(2,1)$, $o(1,2)$ and $o(0,3) \cong o(3)$ (for $\operatorname{sgn} g$ equal to $(+++)$, $(++-)$, $(+--)$, and $(---)$ respectively).

4. Other basic properties of the endomorphism field

The fundamental property of the Lie endomorphism field is that of Theorem 2.1 (Eq. 2.4), which can be expressed in a purely geometric way:

$$[\mathcal{A}, \mathcal{A}] = -2\lambda \,\lrcorner\, \mathcal{A} \tag{4.1}$$

Remark: This expression vanishes for two-step nilpotent algebras, (including Heisenberg-type algebras ([Ka], [FKS])), but, in general, is different from zero, and therefore endomorphism field $\mathcal{A}$ is not integrable. For vector fields of infinitesimal representation, the bi-form (4.1) takes at any point x a vector-value

$$[\mathcal{A}, \mathcal{A}](X_v, X_w) = \mu_x^{-1} \circ \llbracket x, \llbracket \llbracket x, v \rrbracket, \llbracket x, v \rrbracket \rrbracket \rrbracket \tag{4.2}$$

Thus, endomorphism field restricted to an orbit $\mathcal{O} \subset L$ is (locally) integrable if $\llbracket x, \llbracket \llbracket x, v \rrbracket, \rrbracket x, v \rrbracket \rrbracket \rrbracket = 0$ for every $x \in \mathcal{O}$ and every $v, w \in L$. This is true for $so(n)$, $n \leq 4$ and for nilpotent algebras of the upper-triangular $n \times n$ matrices, $n \leq 5$.

Other basic properties of the geometry of a Lie algebra are summarized here:

Corollary 4.1 The endomorphism field on a Lie algebra satisfies:

$$\begin{aligned} (i) &\quad \pounds_J \mathcal{A} = \mathcal{A} \\ (ii) &\quad J \,\lrcorner\, \mathcal{A} = 0 \\ (iii) &\quad \operatorname{Im} \mathcal{A}\big|_{\mathcal{O}} \cong \operatorname{Im} \operatorname{ad}_x^2 \\ (iv) &\quad \operatorname{Ker} \mathcal{A}\big|_{\mathcal{O}} \cong \operatorname{Ker} \operatorname{ad}_x \cap \operatorname{Im} \operatorname{ad}_x \end{aligned} \tag{4.3}$$

Here is a property is analogous to coadjoint representation preserving Kirillov-Poisson structure on the dual Lie algebra.

Proposition 4.2 Endomorphism $\mathcal{A}$ is preserved by the action of the adjoint representation, that is

$$\pounds_{X_v} \mathcal{A} = 0 \qquad \forall \; v \in L \tag{4.4}$$

Proof: Use Leibniz rule to show that $(\pounds_{X_v} A)(w) = 0$ for every w: $(\pounds_{X_v} A)(w) = \pounds_{X_v}(A(w)) - A \pounds_{X_v} w = \pounds_{X_v} X_w - A\llbracket v, w \rrbracket = X_{\llbracket v, w \rrbracket} - X_{\llbracket v, w \rrbracket} = 0$. □

Proposition 4.3 The endomorphism field on a Lie algebra satisfies:

$$\begin{aligned} (iv) &\quad \operatorname{Tr}(\mathcal{A} \circ \mathcal{A}) = K(J, J) \\ (v) &\quad \operatorname{Tr}(\mathcal{A}) = \chi(J) \\ (vi) &\quad K(\mathcal{A}v, w) = -K(v, \mathcal{A}w) \end{aligned} \tag{4.5}$$

where the objects are as follows: K is the Killing form defined for two vectors as $K(v,w) = \operatorname{Tr} \operatorname{ad}_v \circ \operatorname{ad}_w$. When evaluated for (J,J), it becomes a quadratic scalar function $K(J,J) = x^a x^b c^k_{ai} c^i_{bk}$. Similarly, $\chi \in L^*$ is a *characteristic form* on L defined $\chi(v) = \operatorname{Tr} \operatorname{ad}_v$. Property (vi) states that the endomorphism $\mathcal{A}$ is skew-symmetric with respect to the Killing (possibly degenerated) scalar product.

The endomorphism defines for every $k = 1, 2, \ldots,$ a scalar function of the power trace

$$I_k = \operatorname{Tr} \mathcal{A}^k \tag{4.6}$$

Corollary 4.4 Differentials of the trace functions are among the annihilators of $\mathcal{A}$, i.e.,

$$\mathcal{A} \lrcorner \, dI_k = 0 \tag{4.7}$$

5. The endomorphism field and dynamical systems

The dual Lie algebra L^* has deep connections with classical mechanics, one can thus expect that so does the endomorphism field $\mathcal{A}$. We shall discuss three applications:

1. Lax equation.
2. Lagrange mechanics generalized.
3. Generalization of Euler's top equation.

For the first application, we shall prove that "Lax vector fields" form a closed subalgebra under vector field commutator. We shall also define a new "Poisson bracket" on the space of vector fields on Lie algebra. This can be generalized to what we shall call Lax manifold.

As to Lagrange equation, we shall identify $\mathcal{A}$ as a "tangent structure" necessary to define Lagrange equations. This requires a generalization of the notion of Lagrange mechanics.

As to Euler top, we shall see how one can write Euler-Lagrange equations within our approach.

6. The algebra of Lax vector fields

Consider the underlying linear space L of a Lie algebra $\{L, [\![\ ,\ ]\!]\}$ as a manifold. Any smooth vector field B can be viewed as a generator (or "potential") of a dynamical vector field X_B defined

$$X_B = \mathcal{A} B \tag{6.1}$$

The integral curves of X_B satisfy the Lax equations, which in a somewhat imprecise way are expressed

$$\dot{x}(t) = [x, B_x]$$

where the x on the left side is understood as a point in L, while the x on the right side as a vector in L. More accurate (but less readable) version is

$$\begin{aligned}\dot{c}(t) &= [J_{c(t)}, B_{c(t)}] \\ &= \mathcal{A} \circ B \circ c(t)\end{aligned}$$

Definition: Vector fields on a Lie algebra L of form (6.1) will be called **Lax vector fields**, or Lax dynamical systems, and the space of Lax vector fields will be denoted by $\mathcal{X}_{\mathcal{L}}L \cong \mathcal{A}(\mathcal{X}L) \subset \mathcal{X}L$.

A simple and a well-known fact is the existence of Casimir invariants:

Corollary 6.1 Dynamical system defined by vector field (6.1) leaves Casimir functions on L

$$I_k = \operatorname{Tr} \mathcal{A}^k \tag{6.2}$$

invariant, $X_B I_k = 0$, for any $B \in \mathcal{X}L$.

Now, we shall show the geometric meaning of the fundamental property of the endomorphism field in the current context. Namely, it implies that the space of Lax vector fields $\mathcal{X}_{\mathcal{L}}L$ is closed under the commutator of vector fields $[\mathcal{X}_{\mathcal{L}}L, \mathcal{X}_{\mathcal{L}}L] \subset \mathcal{X}_{\mathcal{L}}L$. Thus it forms a Lie subalgebra of all smooth vector fields , $\mathcal{X}_{\mathcal{L}}L < \mathcal{X}L$. A new bracket of vector fields is implied.

Theorem 6.2 If X_B and X_C are two (global) Lax vector fields, then their commutator is a Lax vector field with potential

$$\{B, C\} =: -[\![B, C]\!] + [X_B, C] + [B, X_C] - X_{[B,C]} \tag{6.3}$$

so that there is an homomorphism between the Lax vector fields with the regular vector field commutator and all vector fields with $\{\ ,\ \}$ product:

$$[X_B, X_C] = X_{\{B,C\}} \tag{6.4}$$

Proof: This follows from the fact that $[\mathcal{A}, \mathcal{A}]$ is proportional to $\mathcal{A}$. Rewrite the definition of the Nijenhuis bracket (3.7) for $\mathcal{A}$ and use Theorem 2.1:

$$\begin{aligned}[X_B, X_C] &= [\mathcal{A}B, \mathcal{A}C] \\ &= \frac{1}{2}[\mathcal{A}, \mathcal{A}](B, C) + \mathcal{A}[\mathcal{A}B, C] + \mathcal{A}[B, \mathcal{A}C] - \mathcal{A}^2[B, C] \\ &= -\mathcal{A}[\![B, C]\!] + \mathcal{A}[X_B, C] + \mathcal{A}[B, X_C] - \mathcal{A}X_{[B,C]} \\ &= \mathcal{A}\ \left(-[\![B, C]\!] + [X_B, C] + [B, X_C] - X_{[B,C]} \right)\end{aligned}$$

and, thus, the commutator is of the form (6.1). □

Remark: For two constant vector fields $\tilde{v}$ and $tildew$ extending vectors v and w, it is $\{\tilde{v}, \tilde{w}\} = \widetilde{\llbracket v, w\rrbracket}$. Thus the bracket formula (6.4) reduces in this case to the infinitesimal representation $[X_v, X_w] = X_{\llbracket v,w\rrbracket}$.

Proposition 6.3 The bracket (6.3) can be calculated by the following formula

$$\{A, B\} = \llbracket A,\ B\rrbracket + \underbrace{X_A B - X_B A}_{(A,B)} \tag{6.5}$$

where $X_A B = x^i A^j\ c^k_{ij}\ \partial_k B^p\ \partial_p$.

Proof: (direct calculations; see also proof of Prop. 6.4).

Remark: Notice that although the two right-most terms are defined in coordinates, their difference has a coordinate-free meaning, as it can be defined by $X_A B - X_B A = \{A, B\} - \llbracket A,\ B\rrbracket$.

The bracket $\{\ ,\ \}$ turns the space of vector fields on L into a Lie algebra and can be viewed as a "differential deformation" of the Lie algebra bracket $\llbracket\ ,\ \rrbracket$.

Theorem 6.4: The pair $\{\mathcal{X}L,\ \{\,\cdot\ ,\ \cdot\,\}\}$ forms a Lie algebra, i.e., the bracket (6.3, 5) of vector fields satisfies the following properties:

$$\begin{array}{rl} (i) & \text{(linearity)} \\ (ii) & \{A, B\} = -\{B, A\} \qquad \text{(skewsymmetry)} \\ (iii) & \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \quad \text{(Jacobi identity)} \end{array} \tag{6.6}$$

Proof: If $X, Y \in \mathcal{X}L$ are two vector fields, then we denote $X \triangleright Y = X^i(\partial_i Y^j)\partial_j$ a vector field calculated in linear coordinate system. Thus, formula (6.5) can be written as

$$\{A, B\} = \llbracket A,\ B\rrbracket + (A,\ B)$$

where

$$(A,\ B) = X_A \triangleright B - X_B \triangleright A$$

Now, using the formula $X_A = \llbracket x,\, A \rrbracket$, we get

$$\begin{aligned}
\{\{A,\, B\},\, C\} &= \{\llbracket A,\, B \rrbracket + (A,\, B),\, C\} \\
&= \underbrace{\llbracket \llbracket A,\, B \rrbracket,\, C \rrbracket}_{(a)} + \underbrace{\llbracket (A,\, B),\, C \rrbracket}_{(b)} + \underbrace{(\llbracket A,\, B \rrbracket,\, C)}_{(c)} + \underbrace{((A,\, B),\, C)}_{(d)} \\
&= \underbrace{\llbracket \llbracket A,\, B \rrbracket,\, C \rrbracket}_{(0)} \qquad (a) \\
&+ \underbrace{\llbracket \llbracket x,\, A \rrbracket \triangleright B,\, C \rrbracket}_{(1)} - \underbrace{\llbracket \llbracket x,\, B \rrbracket \triangleright A,\, C \rrbracket}_{(2)} \qquad (b) \\
&+ \underbrace{\llbracket x,\, \llbracket A,\, B \rrbracket \rrbracket \triangleright C}_{(5)} - \underbrace{\llbracket \llbracket x,\, C \rrbracket \triangleright A,\, B \rrbracket}_{(1)} - \underbrace{\llbracket A,\, \llbracket x,\, C \rrbracket \triangleright B \rrbracket}_{(2)} \qquad (c) \\
&+ \underbrace{\llbracket x,\, \llbracket x,\, A \rrbracket \triangleright B \rrbracket \triangleright C}_{(3)} - \underbrace{\llbracket \llbracket x,\, C \rrbracket,\, A \rrbracket \triangleright B}_{(5)} - \underbrace{\llbracket x,\, \llbracket x,\, C \rrbracket \triangleright A \rrbracket \triangleright B}_{(3)} \qquad (d) \\
&- \underbrace{\llbracket x,\, \llbracket x,\, B \rrbracket \triangleright A \rrbracket \triangleright C}_{(4)} + \underbrace{\llbracket \llbracket x,\, C \rrbracket,\, B \rrbracket \triangleright A}_{(5)} + \underbrace{\llbracket x,\, \llbracket x,\, C \rrbracket \triangleright B \rrbracket \triangleright A}_{(4)} \qquad (d)
\end{aligned} \tag{$*$}$$

where the letters (a), (b), (c), and (d) are used to indicate the origin of terms in the second part of the equation. The sum

$$\{\{A,\, B\},\, C\} + \{\{B,\, C\},\, A\} + \{\{C,\, A\},\, B\}$$

contains every term of Equation (*) in each of the tree cyclic permutations of A,B, and C. The sum of such terms marked by any of the numbers (1) to (4) vanish due to opposite signs. The group of terms produced by (0) and terms produced by (5) vanish, each due to the Jacobi identity of Lie algebra product. □

Corollary 6.5 ("Poisson"): If vector fields B and C are Lax potentials of symmetries of a dynamical system, then so is $\{B, C\}$.

Proof: Use the Jacobi identity for vector fields

$$[X_B,\, \underbrace{[X_C, X]}_{=0}\,] + [X,\, \underbrace{[X_B, X_C]}_{=X_{\{B,C\}}}\,] + [X_C,\, \underbrace{[X, X_B]}_{=0}\,] = 0$$

hence the claim. □

What can be used as a Lax potential? Natural vector fields include constant vector fields,

$$\begin{aligned}\{v, w\} &= [\![v, w]\!]\\ \{X_v, w\} &= X_{[\![v,w]\!]}\\ \{X_v, X_w\} &= X_{\mathcal{A}[\![v,w]\!]} - [\![X_v, X_w]\!]\end{aligned}$$

Remark on the "moment map" for the Lax fields

Suppose a Lax dynamical system has a system of symmetries in terms of m Lax vector fields

$$\{Y_1, Y_2, \ldots, Y_k\}$$

that form a closed system, i.e., the commutator of any pair is a linear combination of them. One can say that there is a representation of an k-dimensional Lie algebra K given in terms of Lax vector fields:

$$K \;\to\; \mathcal{X}_{\mathcal{L}}L : \; e \to X_{[e]}$$

Momentum map is a tensor on L with values in the tensor product of the dual Lie algebra and tangent space. This can be thought as a map

$$\mu : \; L \to K^* \otimes TL$$

(in the sense that for every $x \in L$, $\mu_x \in K^* \otimes T_x L$), so that

$$\mathcal{A}\,\langle\, \mu,\, a\,\rangle = X_{[a]}$$

for any $a \in K$. Existence of such a map is equivalent to commutativity of the following diagram of maps

$$\begin{tikzcd}\mathcal{X}L \arrow[r] & \mathcal{X}_{\mathcal{L}}L \\ & K \arrow[u]\end{tikzcd}$$

7. Lagrange mechanics and the endomorphism field — remark. Lie algebra provides an example of Euler-Lagrange structure. This relationship between Lie algebras and Lagrangian formalism opens questions related to the context of geometric quantization and representation theory known for coadjoint orbits in the Lie co-algebras.

$$\begin{tikzcd}
\begin{array}{c}\text{Hamilton}\\ \text{equations}\end{array} \arrow[r] & \begin{array}{c}\text{cotangent bundle}\\ T^*Q\end{array} \arrow[r] & \begin{array}{c}\text{symplectic}\\ \text{geometry}\end{array} \arrow[r] & \begin{array}{c}\text{KKS theorem}\\ \text{(Lie coalgebras)}\end{array}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\text{Lagrange}\\ \text{equations}\end{array} \arrow[r] & \begin{array}{c}\text{tangent bundle}\\ TQ\end{array} \arrow[r] & \begin{array}{c}\text{endomorphic}\\ \text{geometry}\end{array} \arrow[r] & \begin{array}{c}\text{Theorem 3}\\ \text{(Lie algebras)}\end{array}
\end{tikzcd}$$

Recall that the tangent bundle TQ possesses enough structure so that any ("regular") function L on TQ defines its own symplectic structure ω_L (via symplectic potential θ_L such that $d\theta_l = \omega_L$), granting a Hamiltonian formalism induced by the Lagrangian L. This construction, $L \mapsto \omega_L$, has originally been conceptualized in terms of so-called "fiber derivative" on tangent bundle (see [AM], [Tu2]). Later, it has given a more clear geometric picture: while the cotangent bundle possesses a natural tensor field $\omega \in \Lambda^2 Q$ of symplectic structure, the tangent bundle possesses a natural (1,1)-variant tensor field $S \in \mathcal{T}^{(1,1)}Q$ of endomorphisms on $T(TQ)$, or on $T^*(TQ)$. In the natural coordinates (x^i, v^i) on TQ, this tensor can be expressed as $S = \frac{\partial}{\partial v^i} \otimes dx^i$ (sum over i). Its basic property is $\operatorname{Ker} S = \operatorname{Im} S$ (implying nilpotence $S \circ S = 0$). If L is a function on TQ, then one defines a biform $\omega_L = d \circ S \circ dL$, which for "regular" functions is nondegenerate and therefore forms a symplectic structure

Recently, in a series of papers ([Cr1], [Cr2]), a notion of *almost tangent structure* on a differential manifold M has been introduced, as a tensor $S \in \mathcal{T}_1^1 M$ that satisfies

$$\begin{aligned} (i) \quad & \operatorname{Ker} S = \operatorname{Im} S \qquad (\Rightarrow \quad S \circ S = 0) \\ (ii) \quad & [S, S] = 0 \end{aligned} \tag{1.2}$$

where the second condition (ii) is a generalization of the Schouten-Nijenhuis bracket to "vector-valued differential forms" (see e.g [Tu2] and [YI]), which assures (local) integrability of the distribution $\operatorname{Ker} S$. As a result, one obtains all of the structure of the tangent bundle ($\operatorname{Ker} S$ gives the fibering), however without distinguishing the zero-section.

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