

**Relativistic observer and Maxwell's equations:
an example of a non-principal Ehresmann connection**

Jerzy Kocik

Department of Physics, UIUC, Urbana, IL61801

e-mail: jkocik@physics.uiuc.edu

Abstract: The Ehresmann connection on a fiber bundle that is not compatible with a (possible) Lie group structure is illustrated by the geometry of a general anholonomic observer in the Minkowski space. The 3D split of Maxwell's equations induces geometric terms that are the (generalized) curvature and torque of the connection. The notion of torque is introduced here as a Lie coalgebra-valued endomorphism field and measures the deviation of a connection from being principal.

Key words: generalized Ehresmann connection, Nijenhuis bracket, anholonomic space-time observer, Maxwell equations

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Notation: $\mathcal{X}M$ and ΛM denote the $\mathcal{F}M$ -module of smooth vector fields and of exterior differential forms on manifold M , respectively. $\mathcal{D}^k M$ denotes the set of smooth geometric k -distributions, i.e., fields of k -planes on M .

I. Motivation

The standard description of the *connection* in a fiber bundle relies typically on some *group structure* acting along the fibers. In particular, the connection form assumes values in the corresponding Lie algebra, as does the curvature bi-form. This group-based understanding of connection obscures its geometric meaning and excludes possible applications of this concept in cases where no particular symmetry is distinguished or demanded.

Although a general Ehresmann connection on a fiber bundle on which *no* Lie group structure is defined or considered has already been formulated (see e.g., Refs. 1 and 2). Without a group structure, the connection form is described in terms of an endomorphism field, and the curvature tensor can be reinterpreted in terms of Frölicher-Nijenhuis bracket. Chapter III reviews these concepts in a format applicable to physics. We also introduce a notion of a *torque* of connection that measures the deviation of the connection from being principal.

The concept of a general connection seems interesting and should find application in field theory and other areas of mathematical physics. We illustrate the general connection with the anholonomic observer in the Minkowski space. Also, we describe the induced split of Maxwell's equations in a coordinate-free manner (cf Ref. 3), which is another motivation for these notes. We show that the language of generalized connection allows to preserve the elegance of Maxwell's equations in 3D for anholonomic reference frame via the concepts of “curvature” and “torque” of the connection determined by an observer.

II. Observer in the Lorentzian manifold

Let M be a Lorentzian space-time, i.e. a 4-dimensional pseudo-Riemannian (possibly curved) manifold with metric tensor g of signature $(+---)$. An *observer* is a congruence of time-like curves in M (called sometimes a “perfect fluid”⁷). It should not be confused with an *individual* observer — represented by a single time-like curve in M). An Observer can be equivalently represented by a normalized future-oriented vector field $T \in \mathcal{X}M$, $|T|^2 = g(T, T) = 1$. (See e.g., Ref. 4). The field of directions $\mathcal{T} = \text{span}\{T\}$ will be called *time distribution* on M . Define a one-form

$$\tau = g(T, \cdot) \in \Lambda^1 M \quad (2.1)$$

Distribution $\mathcal{S} =: \text{Ker } \tau$ will be called a *spatial* distribution of observer T . Thus a choice of an observer amounts to fixing a pair of two transversal distributions $\mathcal{T} \in \mathcal{D}^1 M$ and $\mathcal{S} \in \mathcal{D}^3 M$ that are mutually orthogonal and span the tangent space at each point of M

$$\mathcal{T} \perp \mathcal{S} \quad \mathcal{T} \oplus \mathcal{S} = TM \quad (2.2)$$

or, using g , to fixing a pair $\{T, \tau\}$, such that

$$\begin{aligned} \text{TIME} &= \mathcal{T} = \text{span } T \\ \text{SPACE} &= \mathcal{S} = \text{Ker } \tau. \end{aligned} \tag{2.3}$$

If the distribution of local space-hyperplanes \mathcal{S} is integrable, the observer is called *holonomic*; otherwise it is *anholonomic*. Note that, by the Frobenius theorem, the space distribution \mathcal{S} is integrable if $\tau \wedge d\tau = 0$ (which in coordinates becomes a rather index-trashed equation, $g_{ab}T^a\partial_i(g_{jk}T^j) + \text{cycl}(b, i, k) = 0$). In this paper, we shall deal with the most general observer; in particular, we shall see how anholonomic observers perceive the world of electrodynamics.

The pair (T, τ) can be used to construct on space-time an endomorphism field, i.e. (1,1)-variant tensor field

$$\kappa = \tau \otimes T. \tag{2.4}$$

Proposition: *The observer endomorphism field κ satisfies*

$$\begin{aligned} (i) \quad [T, \kappa] &= (\mathcal{L}_T \tau) \otimes T \\ (ii) \quad [\kappa, \kappa] &= -2i_T(\tau \wedge d\tau) \otimes T \end{aligned} \tag{2.5}$$

where the brackets represent Frölicher-Nijenhuis products of vector-valued forms.

PROOF: The definition of the Frölicher-Nijenhuis bracket for any pair of vector-valued forms is given in Appendix A. Here we need special cases: for a vector field T , it is $[T, \kappa] = \mathcal{L}_T \kappa$, and (i) follows directly via the Leibniz rule. In the case of two endomorphism fields (vector-valued 1-forms) K and L , the Frölicher-Nijenhuis bracket $[K, L]$ is a vector-valued biform such that for any two vector fields, X and Y , it is (Refs. 5,6)

$$[K, K](X, Y) = 2[KX, KY] - 2K[KX, Y] - 2K[X, KY] + 2KK[X, Y]. \tag{2.6}$$

Substituting (2.4) leads to (ii) (use Eq. (B.11) of Appendix A). \square

Now, we shall interpret a Lorentzian space-time with an observer as a fiber bundle with a connection. Let \sim denote the equivalence relation of belonging to the same integral curve of \mathcal{T} . Define the *seeming space* of observer T as the three-dimensional manifold of equivalence classes of \sim , i.e., $S = M / \sim$. View space-time M as a fiber bundle over S with a natural projection denoted

$$\pi : M \longrightarrow S \tag{2.7}$$

and equipped with a *connection* given by the ‘horizontal’ distribution \mathcal{S} . The connection form coincides with $\kappa = T \otimes \tau$ defined above. Clearly, $\kappa(\mathcal{S}) = 0$ and $\kappa(T) = T$.

Remark: The fiber bundle (2.7) actually *is* a principal fiber bundle: the group $\{\exp tT\}$ defines action via transport along the integral curves of T . However, this group action is not—in general—congruent with the horizontal distribution; that is, κ is not a principal connection. The magnitude of the deviation from being principal can be measured by the Lie derivative of κ along T ; call it the *torque* of the observer (or the connection κ):

$$\text{Torq}(\kappa) = (\mathcal{L}_T \tau) \otimes T \quad (2.8)$$

Any space-like 3-dimensional manifold in M may be viewed as a section $\Psi : S \rightarrow M$. Then the covariant derivative $\nabla \Psi$ measures ‘incompatibility’ of Ψ as a candidate for a ‘space’ with respect to observer T . In particular, notice that $\nabla_v \Psi = g(T, \Psi_* v)$. The next chapter will allow us to define the torque and the curvature of a connection κ via the Nijenhuis bracket, so that

$$\begin{aligned} \text{Torq}(\kappa) &= [T, \kappa] = (\mathcal{L}_T \tau) \otimes T \\ \text{Curv}(\kappa) &= \frac{1}{2}[\kappa, \kappa] = -i_T(\tau \wedge d\tau) \otimes T \end{aligned} \quad (2.9)$$

These should not be confused with the standard curvature defined for the covariant derivative in the Lorentzian manifold.

Figure 1: An observer in space-time

III. Ehresmann Connection on a general fiber bundle

This chapter contains a brief exposition of the concept of a general Ehresmann connection¹ on a fiber bundle in a group-free context is presented. A concept of connection torque is proposed to account for a connection on a principal fiber bundle that is not group-invariant.

A. Connection form. Consider a fiber bundle $\{\pi : E \rightarrow M\}$ over a n -dimensional manifold M with $\dim E = n + N$. A *connection* on a fiber bundle $\{\pi : E \rightarrow M\}$ is any n -distribution $H \subset \mathcal{D}^n E$

$$H : p \longrightarrow H_p \subset T_p E \quad (3.1)$$

of subspaces complementary to the fibers. Consequently, the connection determines a decomposition of tangent spaces

$$T_p E \cong H_p \oplus V_p \quad (3.2)$$

at each $p \in E$, where $V \in \mathcal{D}^N E$, is the *vertical distribution* defined $V_p = \{v \in T_p E \mid \pi_* v = 0\}$. In particular, each tangent vector $X \in TE$ may be uniquely decomposed into a ‘vertical’ and a ‘horizontal’ part: $X = X_H + X_V$. Once chosen, H (not necessarily integrable) is called the *horizontal distribution*.

Splitting (3.2) can be described in terms of an endomorphism on E , namely the *projection* of tangent vectors of $T_p E$ onto subspace V_p along subspace H_p . This is a linear map $\kappa : T_p E \rightarrow T_p E$ such that

$$\begin{aligned} (i) \quad & \kappa \circ \kappa = \kappa \\ (ii) \quad & \kappa|_H = 0 \quad \text{and} \quad \kappa|_V = \text{id} \end{aligned} \quad (3.3)$$

In particular, $\kappa(X) = X_V$. Operator κ has been called somewhat improperly a ‘connection form,’ although it is a $(1,1)$ -type tensor field and will hence be termed a ‘field of endomorphisms.’

B. Lifting vectors. The horizontal distribution defines the *lifting* of vectors from the basis manifold M to the total space E ; for any point $p \in E$ and $m = \pi(p) \in M$, define a linear map $\tilde{\cdot} : T_m M \rightarrow T_p E$ such that

$$\begin{aligned} (i) \quad & \tilde{v} \in H \\ (ii) \quad & \pi_*(\tilde{v}) = v \end{aligned} \quad (3.4)$$

for any $v \in TM$. This can be extended to vector fields on M , which are lifted to (horizontal) vector fields on E . The definitions may be summarized in terms of the following *exact* sequence

$$0 \longrightarrow T_{\pi(p)}M \xrightarrow{\sim} T_pE \xrightarrow{\kappa} T_pE \xrightarrow{\pi_*} T_{\pi(p)}M \longrightarrow 0. \quad (3.5)$$

The extensions of the two operations (i) and (ii) to the vector fields will be denoted by the same symbols:

$$\begin{array}{ll} (i) & \text{lifting} \quad \tilde{\cdot} : \mathcal{X}M \longrightarrow \mathcal{X}_HE \\ (ii) & \text{projection} \quad \kappa : \mathcal{X}E \longrightarrow \mathcal{X}_VE \end{array}$$

where \mathcal{X}_VE and \mathcal{X}_HE denote *vertical* and *horizontal* vector fields, respectively.

C. Covariant derivative of a section. A horizontal distribution defines uniquely a *covariant derivative* of a section $\Psi : M \rightarrow E$ along vector $v \in T_mM$ as a vertical vector

$$\nabla_v \Psi = \kappa(\Psi^*(v)) \in V_{\Psi(m)} \quad (3.6)$$

at point $\Psi(m) \in \mathcal{E}$. (Equivalently, $\nabla_v \Psi = \Psi_*v - \tilde{v}$, where \tilde{v} is understood as lifted to the points of section $\Psi(M)$.) In general, a *covariant derivative* of section Ψ may be viewed as a linear map $T_mM \rightarrow T_{\Psi(m)}E$ defined

$$\nabla \Psi = \kappa \circ \Psi_*.$$

Figure 2

Connection and covariant derivative

This, clearly, extends to vector fields and gives a (point-wise linear) map

$$\nabla \Psi : \mathcal{X}M \rightarrow \mathcal{X}E \Big|_{\Psi(M)} \quad (3.7)$$

from vector fields on M into vertical vector fields along the image $\Psi(M)$ of the section.

D. Curvature via Nijenhuis bracket. Since the endomorphism κ may be viewed as a vector-valued differential 1-form, the Frölicher-Nijenhuis theory of differential ‘vector forms’ (i.e., $(1, k)$ -type tensors) applies to it. In particular, curvature may be defined in a coordinate-free, geometric fashion.²

Definition: If κ is the (1,1)-tensor of connection, then the curvature is a (1,2)-tensor defined as

$$\Omega = \frac{1}{2}[\kappa, \kappa]. \quad (3.8)$$

where $[\ , \]$ denotes the Nijenhuis bracket of vector-forms (see Appendix A).

It is easy to see that the form Ω is vertical, $i_v\Omega = 0$ for any vertical v . One can thus define a bi-form ω on M that assumes values in the vertical vector fields on E ,

$$\omega(v, w) = \Omega(\tilde{v}, \tilde{w}). \quad (3.9)$$

If $X, Y \in \mathcal{X}M$ are two global vector fields on M , then $\omega(X, Y)$ is a global (vertical) vector field on E .

Since vector-forms form a graded Lie algebra, the Bianchi identity follows immediately²

$$[\Omega, \kappa] = 0. \quad (3.10)$$

Indeed, $[\Omega, \kappa] = [[\kappa, \kappa], \kappa] = 0$ due to the (graded) Jacobi identity. This pleasant result shows that the lack of a particular group structure in the fibers is not an obstruction to the existence of quite a lot of structure in the notion of a general connection.

E. Principal fiber bundle and torque. Let $\pi : E \rightarrow M$ be a principal fiber bundle, i.e., let a group G act freely and transitively on fibers of E . In particular, the Lie algebra L of G is represented by the vertical vector fields

$$\rho : L \longrightarrow \mathcal{X}_v E. \quad (3.11)$$

induced by the diffeomorphisms of the action of the group. The connection κ is *principal* if it is invariant under the action of the group. Consider a general Ehresmann connection on a principal fiber bundle that does not necessarily agree with the group action.

Definition: *Torque* of a general connection on a principal fiber bundle is a tensor

$$\text{Torq}(\kappa) \in L^* \otimes T^{(1,1)} \quad (3.12)$$

which, evaluated on $a \in L$, is

$$\text{Torq}(\kappa)(a) = \mathcal{L}_{\rho(a)}\kappa. \quad (3.13)$$

Torque is clearly linear in a , and measures the deviation of the connection from being principal. If $\{e_p\}$ is a basis in L , and $\{\varepsilon_p\}$ in the dual space L^* , then

$$\text{Torq}(\kappa) = \varepsilon_p \otimes \mathcal{L}_{\rho(e_p)}\kappa \quad (3.14)$$

(summation over p).

IV. Geometry of Maxwell equations

This chapter analyzes the geometry of Maxwell's equations in the context of a (generalized) observer.

A. Maxwell Equations in M^3 and M^4 .

Let us review the geometric content of Maxwell's equations as presented in the standard way (see also Refs. 7, 8).

In the “pre-relativistic” formulation of Maxwell's equations, space is a 3-dimensional manifold M^3 equipped with a Riemannian structure g , while time appears in the theory as a *parameter* rather than a coordinate.

$$\begin{aligned} dE^* &= \varrho & dB &= 0 \\ dE &= -\partial_t B & dB^* &= J + \partial_t E^* \end{aligned} \tag{4.1}$$

where the fields E and B are differential 1-form and 2-form respectively. The Hodge star \star acts as a linear map $\star: \Lambda^k M \rightarrow \Lambda^{3-k} M$ and is defined by

$$\star(\alpha, \beta) = \alpha \wedge \star\beta \tag{4.2}$$

where the bracket denotes the scalar product of forms (of the same degree) induced from g in M^3 . An immediate implication of Maxwell's equations is the ‘continuity equation,’ $dJ + \partial_t \varrho = 0$.

In the relativistic formulation, the Maxwell equations are rewritten in 4-dimensional space-time, a manifold M^4 equipped with a pseudo-Riemannian structure g of signature $(+ - - -)$. An *electromagnetic field* is a differential bi-form $F \in \Lambda^2 M^4$, and the *current-charge density* is a differential 3-form $j \in \Lambda^3 M$. The Maxwell equations are:

$$\begin{aligned} dF &= 0 \\ d * F &= j \end{aligned} \tag{4.3}$$

The first equation implies via the Poincaré Lemma the (local) existence of a differential 1-form $A \in \Lambda^1 M^4$, such that $F = dA$. The second equation implies that $dj = 0$ (continuity equation). The laws of electromagnetism may be summarized in the following diagram:

$$\begin{array}{ccccc}
a & \xrightarrow{d} & F & \xrightarrow{d} & 0 \\
& & \downarrow * & & \\
& & *F & \xrightarrow{d} & j \xrightarrow{d} 0
\end{array} \tag{4.4}$$

where

$$\begin{aligned}
F \in \Lambda^2 M & \leftarrow \text{electromagnetic field} \\
G = *F \in \Lambda^2 M & \leftarrow \text{dual electromagnetic field} \\
A \in \Lambda^1 M & \leftarrow \text{electromagnetic potential} \\
j \in \Lambda^3 M & \leftarrow \text{charge-current}
\end{aligned}$$

The Hodge star $*$ is now defined as a point-wise linear map $\Lambda^k M \rightarrow \Lambda^{4-k} M$, such that

$$* (\alpha, \beta) = \alpha \wedge * \beta \tag{4.5}$$

for any two forms of the same degree, where $(,)$ denotes the pseudo-Euclidean scalar product on M^4 .

B. An observer and fields

Let A and α be a vector field and a differential 1-form, respectively, $\alpha \in \Lambda^1 M$ and $A \in \mathcal{X}^1 M$. Then for any k -form $\omega \in \Lambda^k M$, the following formula holds (see e.g., Ref. 9):

$$A \lrcorner (\alpha \wedge \omega) + \alpha \wedge (A \lrcorner \omega) = \alpha(A) \cdot \omega. \tag{4.6}$$

Denoting the *interior* and *exterior* multiplications by

$$\begin{aligned}
i_A \omega &= A \lrcorner \omega = \omega(A, \dots) \\
e_\alpha \omega &= \alpha \wedge \omega
\end{aligned}$$

we may rewrite (4.6) as $i_X \circ e_\alpha + e_\alpha \circ i_A = \langle \alpha, X \rangle \cdot \text{id}$. If form α and vector A are chosen so that $\alpha(A) = 1$, then one has a ‘decomposition formula’

$$i_X \circ e_\alpha + e_\alpha \circ i_A = \text{id} \tag{4.7}$$

that allows an exterior form to be split into two parts—one that is “parallel” to the direction of A and one that is “parallel” to the complementary direction of the co-plane spanned by the kernel of α , $\text{Ker } \alpha$.

Now, consider a Lorentzian space-time with an observer $\{T, \tau\}$, with $\tau(T) = 1$. The split formula

$$e_\tau \circ i_T + i_T \circ e_\tau = \text{id}$$

allows any exterior form ω in space-time to be split into a sum of two terms characteristic for a particular observer, i.e.

$$\begin{aligned} \omega &= e_\tau \circ i_T \omega + i_T(\tau \wedge \omega) = \omega_{time} + \omega_{space} \\ &= \tau \wedge \omega_T + \omega_S \end{aligned} \tag{4.8}$$

where both ω_T and ω_S are purely spatial, since they vanish under i_T . Applying the split to the bi-form F of an electromagnetic field gives the observers' electric and magnetic component:

$$F = F_e + F_m \quad \text{where} \quad \begin{cases} F_e = e_\tau \circ i_T F \equiv \tau \wedge (T \lrcorner F) \\ F_m = i_T \circ e_\tau F \equiv T \lrcorner (\tau \wedge F). \end{cases}$$

The observer perceives the magnetic field defined as a bi-form $B = i_T(\tau \wedge F)$ and the electric field defined as 1-form $E = -i_T F$, and the original tensor can be reconstructed as $F = E \wedge \tau + B$. A similar split occurs for other fields, as summarized below:

$F = E \wedge \tau + B$	where	$E = -i_T F$	$B = i_T \tau \wedge F$
$G = \tau \wedge H + D$	where	$H = i_T G$	$D = i_T \tau \wedge G$
$j = \varrho - \tau \wedge J$	where	$J = -i_T j$	$\varrho = i_T \tau \wedge j$
$a = \varphi \tau + A$	where	$\varphi = i_T a$	$A = i_T \tau \wedge a$

C. An observer and the Maxwell equations

Now we shall see the Maxwell equations under the geometric split (4.8). First, however, a definition of the two operators;

Definition: *The spatial exterior derivative d_3 and the time derivative of a differential form are, for observer $\{T, \tau\}$, the following operators:*

$$\begin{aligned} \dot{\omega} &= \mathcal{L}_T \omega \\ d_3 \omega &= i_T \tau \wedge d\omega. \end{aligned} \tag{4.9}$$

Lemma 1. *In the reference system of observer $\{T, \tau\}$, a differential equation $d\omega = \sigma$ involving exterior forms splits into a pair of equations*

$$\begin{aligned} d_3 \omega_S &= \sigma_S + \text{Curv}(\tau) \lrcorner \omega \\ -\dot{\omega}_S + d_3 \omega_T &= \sigma_T - \text{Torq}(\tau) \lrcorner \omega \end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\text{Torq}(\tau) \lrcorner \omega &= (\mathcal{L}_T \tau) \wedge i_T \omega = \dot{\tau} \wedge \omega_T \\ \text{Curv}(\tau) \lrcorner \omega &= -i_T(\tau \wedge d\tau) \wedge i_T \omega = -i_T(\tau \wedge d\tau) \wedge \omega_T.\end{aligned}\tag{4.11}$$

The first equation of (4.10) is purely spatial (vanishes under i_T), and the other is temporal (vanishes under e_τ).

Proof: Consider $d(\tau \wedge i_T \omega + i_T \tau \wedge \omega) = \sigma$. Apply $i_T \wedge \tau$ to both sides to obtain the first equation:

$$\begin{aligned}\sigma_S &= i_T(\tau \wedge d(\tau \wedge i_T \omega + i_T \tau \wedge \omega)) \\ &= i_T(\tau \wedge (d\tau \wedge \omega_T - \tau \wedge d\omega_T + d\omega_S)) \\ &= i_T(\tau \wedge d\tau) \wedge \omega_T + i_T(\tau \wedge d\omega_S) \\ &= -\text{Curv} \lrcorner \omega_T + d_3 \omega_S\end{aligned}$$

To obtain the second equation, apply i_T to both sides:

$$\begin{aligned}\sigma_T &= i_T d(\tau \wedge i_T \omega + i_T \tau \wedge \omega) \\ &= i_T(d\tau \wedge \omega_T - \tau \wedge d\omega_T + d\omega_S) \\ &= (i_T d\tau) \wedge \omega_T - i_T(\tau \wedge d\omega_T) + i_T d\omega_S \\ &= (\mathcal{L}_T \tau) \wedge \omega_T - d_3 \omega_T + \dot{\omega}_S\end{aligned}$$

where we used $\mathcal{L}_T = i_T d + di_T$. \square

Theorem 1. Maxwell's equations in the reference system of observer $\{T, \tau\}$ are

$$\begin{aligned}d_3 E &= -\dot{B} - \text{Torq}(\tau) \lrcorner F & d_3 H &= \dot{D} + J + \text{Torq}(\tau) \lrcorner G \\ d_3 B &= \text{Curv}(\tau) \lrcorner F & d_3 D &= \rho + \text{Curv}(\tau) \lrcorner G\end{aligned}\tag{4.12}$$

where

$$\begin{aligned}\text{Torq}(\tau) \lrcorner \omega &= (\mathcal{L}_T \tau) \wedge i_T \omega \\ \text{Curv}(\tau) \lrcorner \omega &= -i_T(\tau \wedge d\tau) \wedge i_T \omega.\end{aligned}\tag{4.13}$$

Proof: Apply Lemma 1 to both equations of (4.3). \square

From (4.12) follows that $\text{Curv}(\tau) \lrcorner F$ can be interpreted as an “apparent” magnetic charge, and $\text{Torq}(\tau) \lrcorner F$ as an “apparent” magnetic current. Similarly, $\text{Curv}(\tau) \lrcorner G$ and

$\text{Torq}(\tau) \lrcorner G$ contribute to the effective electric charge and current, respectively. Clearly, if $d\tau = 0$, then (4.12) reduces to the standard 3D Maxwell equations.

The continuity equation and the potential can be treated similarly.

Corollary 2. *The continuity equation for a general observer becomes*

$$\dot{\rho} + d_3 J = -\text{Torq}(\tau) \lrcorner j \quad (4.14)$$

Proof: Write $j = \tau \wedge (-J) + \rho$ and apply Lemma 1. There is only one 3D equation, since dj is a 4-form and $\tau \wedge$ kills it. \square

Corollary 3. *The equations for a potential are, for a general observer*

$$\begin{aligned} E &= -\dot{A} + d_3 \varphi - \text{Torq} \lrcorner a \\ B &= d_3 A - \text{Curv} \lrcorner a. \end{aligned} \quad (4.15)$$

Proof: Write $F = \tau \wedge (-E) + B$ and apply Lemma 1. \square

Definition: *The reduced (spatial) Hodge star for an observer (T, τ) is defined as*

$$*_3 = i_T \circ * \quad (4.16)$$

where $*$ denotes the Hodge star in space-time.

Remark: If $d\tau = 0$ and $\iota : S \subset M$ is a 3-dimensional submanifold such that $\iota^* \tau = \text{const}$ (“instantaneous space”), then the reduced spatial Hodge star (6.16) coincides with the Hodge star \star on S , determined by the induced metric $\iota^* g$, that is, $\star \circ \iota^* = \iota^* \circ *_3$.

Proposition 4. *The Hodge star $*$ intertwines the operators of e_τ and i_T :*

$$\begin{aligned} i_T \circ * &= (-)^k * \circ e_\tau \\ e_\tau \circ * &= (-)^{k+1} * \circ i_T \end{aligned} \quad (4.17)$$

where k is the degree of the form on which the product acts, and we abbreviate $(-)^k = (-1)^k$.

Proof: Let us show the first relation:

$$\begin{aligned} * \circ e_\tau \omega &= *(\tau \wedge \omega) = (g^{-1}(\tau \wedge \omega)) \lrcorner \eta \\ &= (T \wedge \Omega) \lrcorner \eta = (-1)^k (\Omega \wedge T) \lrcorner \eta \\ &= (-)^k i_T \circ i_\Omega \eta = (-)^k i_T * \omega \end{aligned}$$

where $\Omega = g^{-1}(\omega)$ and $k = \deg \omega$. The second relation follows similarly. \square

Corollary 5. *For any observer (T, τ) , the Hodge relation $*F = G$ reduces to*

$$*_3 E = D \quad \text{and} \quad *_3 H = B \quad (4.18)$$

Proof: Indeed, using (4.14), one calculates $*_3 E = i_T * E = -i_T * i_T F = i_T e_\tau * F = i_T e_\tau G = D$. Similarly for a magnetic field, $*_3 B = i_T * B = i_T * i_T e_\tau F = i_T e_\tau * e_\tau F = i_T e_\tau i_T * F = i_T * F = i_T G = H$. \square

Note that the operators i_T and $e_\tau = \tau \wedge$ satisfy the interesting algebraic property

$$i_T \circ e_\tau \circ i_T = i_T \quad \text{and} \quad e_\tau \circ i_T \circ e_\tau = e_\tau \quad (4.19)$$

(cf., Temperley-Lieb algebra or Artin braid group).

Appendix A

Nijenhuis bracket

Consider endomorphisms of the space of exterior differential forms. A graded map $a : \Lambda M \rightarrow \Lambda M$ is of degree $\deg a \in \mathbb{Z}$, if $a : \Lambda^k M \rightarrow \Lambda^{k+\deg a} M$. Define the following product of graded maps in ΛM (see e.g., Refs. 10, 11, 12, 6):

$$\llbracket a, b \rrbracket = a \circ b - (-)^{(\deg a \cdot \deg b)} b \circ a \quad (A.1)$$

and extend it by linearity to the space spanned by graded maps of ΛM . These immediate properties

- (i) (bi-linearity)
- (ii) $\llbracket a, b \rrbracket = -(-)^{(\deg a \cdot \deg b)} \llbracket b, a \rrbracket$ (super skew-symmetry) (A.2)
- (iii) $(-)^{(\deg a \cdot \deg c)} \llbracket a, \llbracket b, c \rrbracket \rrbracket + \text{cyclic terms} = 0$ (Jacobi identity)

turn the space of graded maps into a Lie superalgebra with superbracket $\llbracket \cdot, \cdot \rrbracket$. Endomorphism D is called a *derivation*, if it satisfies the graded Leibniz rule

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-)^{\deg D \cdot \deg \alpha} \alpha \wedge D\beta \quad (A.3)$$

The well-known examples include the exterior derivative and the contraction with a vector field

$$\Lambda^k M \xrightarrow{d} \Lambda^{k+1} M \quad \text{and} \quad \Lambda^k M \xrightarrow{i_X} \Lambda^{k-1} M \quad (A.4)$$

where $X \in \mathcal{X}M$. Another derivation can be defined by contraction with any vector-form (vector-valued form), i.e., an element of

$$\mathcal{X}^1 M \otimes \Lambda^k M \subset T^{1,k} M.$$

Indeed, for a homogeneous vector-form

$$A = \alpha \otimes A \quad \alpha \in \Lambda^k M \quad A \in \mathcal{X}M$$

the contraction defined as

$$i_A \omega = \alpha \wedge i_A \omega \quad (A.5)$$

is a derivation of degree $(k-1)$, which can be extended by linearity to a general vector-form.

The set $Der \Lambda M$ of all derivations is closed under the product of the superbracket. Define a Lie derivative along a vector-form A as

$$\mathcal{L}_A = \llbracket i_A, d \rrbracket. \quad (A.6)$$

Since derivations form a superalgebra, \mathcal{L}_A is a derivation. By the same argument, the superbracket of two Lie derivatives is a derivation. It can be shown that for any two vector-forms A and B , there exists a vector-form denoted $[A, B]$, such that

$$\llbracket \mathcal{L}_A, \mathcal{L}_B \rrbracket = \mathcal{L}_{[A, B]} \quad (A.7)$$

The vector-form $[A, B]$ is called a Nijenhuis (or Frölicher-Nijenhuis) bracket of A and B . The following theorem generalizes this result⁶:

Theorem. (Michor) *Any derivation $D \in Der_k \Lambda M$ is of form*

$$D = \mathcal{L}_K + i_L \quad (A.8)$$

for some vector-forms K and L . In particular, the exterior derivative is $d = \mathcal{L}_{\text{id}}$. In addition

- (i) $L = 0$ iff $\llbracket D, d \rrbracket = 0$
- (ii) $K = 0$ iff $D|_{\mathcal{F}M} = 0$.

Since $\llbracket \mathcal{L}_K, \mathcal{L}_L \rrbracket, d = 0$ (by the Jacobi identity (A.2.iii)), then, by the above theorem, there exists a vector-form $[K, L]$ such that

$$\llbracket \mathcal{L}_K, \mathcal{L}_L \rrbracket = \mathcal{L}_{[K, L]}.$$

For the special cases where K and L are two endomorphism fields (vector-valued 1-forms), the Nijenhuis bracket $[K, L]$ is a vector-valued biform, which, evaluated for two vector fields X and Y , is

$$\begin{aligned} [K, L](X, Y) &= [KX, LY] - [KY, LX] \\ &\quad - L[KX, Y] + L[KY, X] \\ &\quad - K[LX, Y] + K[LY, X] \\ &\quad + LK[X, Y] + KL[X, Y]. \end{aligned} \tag{A.9}$$

The Nijenhuis torsion of an endomorphism field K is defined via the Nijenhuis bracket of K with itself:

$$N_K = \frac{1}{2}[K, K](X, Y) = [KX, KY] - K[KX, Y] - K[X, KY] + K^2[X, Y]. \tag{A.10}$$

The Nijenhuis torsion of a homogeneous tensor $A \otimes \alpha$ is

$$\begin{aligned} \frac{1}{2}[A \otimes \alpha, A \otimes \alpha] &= A \otimes (\alpha \wedge \mathcal{L}_A \alpha - \alpha(A) d\alpha) \\ &= A \otimes (\alpha \wedge d(\alpha(A)) - i_A \alpha \wedge d\alpha). \end{aligned} \tag{A.10}$$

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