FOUR PROOFS OF A GENERALIZATION OF THE DESCARTES CIRCLE THEOREM

J. B. WILKER, University of Toronto

1. Introduction. In a recent paper, H. S. M. Coxeter [3] supplies a simplified version of Beecroft's proof [1] of the Descartes circle theorem ([4], pp. 37–50)

$$2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2) = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^2,$$

where ϵ_i ($i=1, 2, 3, 4$) are the bends (reciprocals of the radii) of four circles in mutual external contact. If the first three circles bound a curvilinear triangle containing the fourth, the relation may be used in Steiner's unsymmetric form ([7], p. 274) as a formula for the fourth bend

$$\epsilon_4 = \epsilon_1 + \epsilon_2 + \epsilon_3 + 2\sqrt{\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1}.$$

As Coxeter indicates, the number $\delta = \sqrt{\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_3\epsilon_1}$ is geometrically meaningful. It is the bend of the circumcircle of the curvilinear triangle determined by the circles of bend ϵ_1, ϵ_2 and ϵ_3. With this change of notation, Steiner's formula assumes the more attractive form

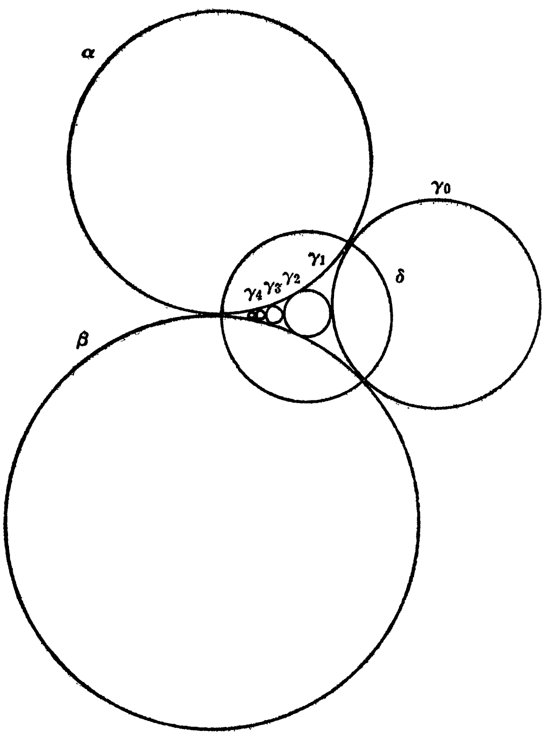

FIG. 1

$$\epsilon_4 = \epsilon_1 + \epsilon_2 + \epsilon_3 + 2\delta.$$

For the generalization to be discussed below, it is convenient to adopt the notation $\epsilon_1=\alpha$, $\epsilon_2=\beta$, $\epsilon_3=\gamma_0$, $\epsilon_4=\gamma_1$. Then it is natural to write γ_n for the bend of the nth circle defined inductively as tangent to the circles of bend α, β and γ_{n-1}, (Figure 1). The generalized formula is

$$\gamma_n = \gamma_0 + 2n\delta + n^2(\alpha + \beta).$$

Four proofs will be given for this formula. The first two, due respectively to Z. A. Melzak [5] and R. G. Stanton et al. [8], begin with Steiner's formula. Melzak iterates it n times by the method of conjugate functions; Stanton solves a related difference equation. The third proof, kindly supplied to me by H. S. M. Coxeter, uses two applications of the Descartes circle theorem to get a linear relation among the bends which gives the result. The final proof involves inversion and gives the generalized formula immediately with Steiner's result appearing as the special case $n=1$.

2. Proofs beginning with Steiner's formula. Applying Steiner's formula to the curvilinear triangle bounded by circles of bend α, β and γ_{n-1}, we find

$$\begin{aligned}\gamma_n &= \alpha + \beta + \gamma_{n-1} + 2\sqrt{\alpha\beta + (\alpha + \beta)\gamma_{n-1}}\\ &= a + \gamma_{n-1} + 2\sqrt{b + a\gamma_{n-1}},\end{aligned}$$

where $\alpha+\beta=a$ and $\alpha\beta=b$.

(i) *Version of Melzak.* Let $f(x)=a+x+2\sqrt{b+ax}$. Then $\gamma_n=f(\gamma_{n-1})$ and $\gamma_n=f_n(\gamma_0)$ where f_n is the nth iterate of f.

Let $g(x)=x+1$ and $h(x)=\sqrt{b+ax}/a$. Then $h^{-1}(x)=ax^2-b/a$ and

$$\begin{aligned}h^{-1}\circ g\circ h(x) &= h^{-1}\circ g(\sqrt{b + ax}/a)\\ &= h^{-1}(\sqrt{b + ax}/a + 1)\\ &= (b + ax)/a + 2\sqrt{b + ax} + a - b/a\\ &= f(x).\end{aligned}$$

It follows that $f_n=h^{-1}\circ g_n\circ h$. But $g_n(x)=x+n$ and a computation analogous to the above gives $f_n(x) = x + 2n\sqrt{b + ax} + n^2a$. Hence

$$\begin{aligned}\gamma_n &= f_n(\gamma_0)\\ &= \gamma_0 + 2n\sqrt{\alpha\beta + (\alpha + \beta)\gamma_0} + n^2(\alpha + \beta)\\ &= \gamma_0 + 2n\delta + n^2(\alpha + \beta).\end{aligned}$$

(ii) *Version of Stanton et al.* Let $\gamma_n=ac_n^2-b/a$. Then $c_n=\sqrt{b+a\gamma_n}/a$ and

$$\begin{aligned}c_{n+1}^2 - c_n^2 &= (\gamma_{n+1} - \gamma_n)/a\\ &= 1 + 2\sqrt{b + a\gamma_n}/a\\ &= 1 + 2c_n.\end{aligned}$$

Hence $c_{n+1}^2=(c_n+1)^2$ and $c_{n+1}=c_n+1$. It follows that

$$c_n = c_0 + n = \sqrt{b + a\gamma_0}/a + n$$

and

$$\begin{aligned}\gamma_n &= ac_n^2 - b/a \\ &= b/a + \gamma_0 + 2n\sqrt{b + a\gamma_0} + n^2a - b/a \\ &= \gamma_0 + 2n\sqrt{\alpha\beta + (\alpha + \beta)\gamma_0} + n^2(\alpha + \beta) \\ &= \gamma_0 + 2n\delta + n^2(\alpha + \beta).\end{aligned}$$

3. Proof beginning with the Descartes circle theorem. At the ith stage we have four circles of bend α, β, γ_{i-1} and γ_i in mutual external contact. By the Descartes circle theorem,

$$2(\alpha^2 + \beta^2 + \gamma_{i-1}^2 + \gamma_i^2) = (\alpha + \beta + \gamma_{i-1} + \gamma_i)^2.$$

Similarly, from the $(i-1)$th stage,

$$2(\alpha^2 + \beta^2 + \gamma_{i-2}^2 + \gamma_{i-1}^2) = (\alpha + \beta + \gamma_{i-2} + \gamma_{i-1})^2.$$

Subtracting and dividing by $\gamma_i-\gamma_{i-2}$,

$$2(\gamma_i + \gamma_{i-2}) = 2(\alpha + \beta) + \gamma_{i-2} + 2\gamma_{i-1} + \gamma_i.$$

Rearranging and writing the resulting equation for $i=n, n-1, \cdots, 2$,

$$\begin{aligned}2(\alpha + \beta) - \gamma_{n-2} + 2\gamma_{n-1} - \gamma_n &= 0, \\ 2(\alpha + \beta) - \gamma_{n-3} + 2\gamma_{n-2} - \gamma_{n-1} &= 0, \\ \cdots\cdots\cdots\cdots\cdots&, \\ 2(\alpha + \beta) - \gamma_0 + 2\gamma_1 - \gamma_2 &= 0.\end{aligned}$$

Multiplying these equations by $1, 2, \cdots, n-1$ and adding, we obtain

$$n(n - 1)(\alpha + \beta) - (n - 1)\gamma_0 + n\gamma_1 - \gamma_n = 0,$$

whence

$$\gamma_n = \gamma_0 + n(-\alpha - \beta - \gamma_0 + \gamma_1) + n^2(\alpha + \beta).$$

The fact that $-\alpha-\beta-\gamma_0+\gamma_1=2\delta$ is part of the simplified Beecroft proof mentioned earlier (equation 2.3 of [**3**], with $-\delta$ for η_1).

4. Proof using inversion. Two simple lemmas of Morley [6] are needed.

4.1 *A line $D>0$ units from the center of inversion inverts in a circle of radius K to a circle of bend $2D/K^2$.*

4.2 *A circle of radius R centered $D>R$ units from the center of inversion inverts in a circle of radius K to a circle of bend $(D^2-R^2)/K^2R$.*

We proceed from these lemmas with a method analogous to that used by Coxeter [2] in giving several proofs of the Descartes circle theorem.

Inversion in a circle of radius K centered at (x_0, y_0), the point of intersection of the circles of bend α and β, transforms these circles into parallel lines and takes the circle of bend δ into a perpendicular line. For a proper choice of K and (x_0, y_0) these lines may be taken to be $y=\pm 1$ and $x=0$ respectively. Then the circles of bend $\gamma_n (n=0, 1, 2, \cdots)$ invert to the congruent circles

$$(x-2n)^2+y^2-1=0$$

between the parallel lines, (Figure 2).

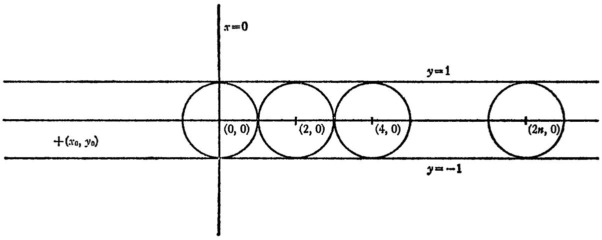

FIG. 2

As inversion is a transformation of period two, the same inversion restores the original figure. By 4.1,

$$\alpha=\frac{2(1-y_0)}{K^2}, \quad \beta=\frac{2(1+y_0)}{K^2}, \quad \delta=\frac{-2x_0}{K^2}.$$

By 4.2,

$$\begin{aligned}\gamma_n &= \frac{(x_0-2n)^2+y_0^2-1}{K^2}\\ &= \frac{x_0^2+y_0^2-1}{K^2}-\frac{4nx_0}{K^2}+\frac{4n^2}{K^2}\\ &= \gamma_0+2n\delta+n^2(\alpha+\beta).\end{aligned}$$

To retrieve the traditional form of Steiner's formula we compute

$$\alpha\beta+(\alpha+\beta)\gamma_0=\frac{4(1-y_0^2)}{K^4}+\frac{4}{K^2}\,\frac{x_0^2+y_0^2-1}{K^2}=\frac{4x_0^2}{K^4}=\delta^2.$$

References

1. Philip Beecroft, Properties of circles in mutual contact, Lady's and Gentleman's Diary, (1842) 91–96.
2. H. S. M. Coxeter, Loxodromic sequences of tangent spheres, Equationes Mathematicae, 1 (1968) 104–121.

3. H. S. M. Coxeter, The problem of Apollonius, this MONTHLY, 75 (1968) 5–15.
4. René Descartes, Œuvres, vol. 1, C. Adam and P. Tannery, Paris, 1901.
5. Z. A. Melzak, Infinite packings of disks, Canad. J. Math., 18 (1966) 838–852.
6. Frank Morley, The hexlet, Nature, 139 (1937) 72.
7. Jacob Steiner, Fortsetzung der geometrischen Betrachtungen, J. für Math., 1 (1826) 252–288.
8. R. G. Stanton, H. C. Williams, C. R. Zarnke, A packing problem, Canad. Math. Bull., 10 (1967) 287–297.

A KREIN-MILMAN THEOREM FOR PARTIALLY ORDERED SETS

KIRBY A. BAKER, California Institute of Technology

The familiar Krein-Milman Theorem for locally convex topological linear spaces T states that any compact convex subset S of T is the closed convex hull of the set of its own extreme points [5, Th. 11.5, p. 138]. (It will be recalled that p is an extreme point of S when closed line segments lying in S can contain p only as an end point.)

For partially ordered sets there is also a notion of convexity, in which intervals play the rôle of line segments. We shall give an analogue of the Krein-Milman theorem for this case.

If S is a subset of a partially ordered set P, the *convex hull* of S consists of all $x\in P$ such that $s\leqq x\leqq t$ for some $s, t\in S$. S is a *convex* subset of P if S is its own convex hull. Equivalently, S is convex if, whenever $s, t\in S$ and $s\leqq t$, S must contain the whole interval $[s, t]=\{x\in P: s\leqq x\leqq t\}$ [1, p. 7].

If S is convex, it is natural to call $c\in S$ an *extreme point* of S if for any interval $[s, t]\subseteq S$, $c\in[s, t]$ implies $c=s$ or $c=t$. More simply, c is an extreme point of S if c is a minimal or a maximal element of S.

To express the condition of compactness, it is necessary to choose a topology on P. A reasonable requirement to impose is that for each $x\in P$, the "(closed) rays" $J(x)=\{y\in P: y\geqq x\}$ and $M(x)=\{y\in P: y\leqq x\}$ be closed sets. We shall choose the weakest topology meeting this requirement, namely the Frink interval topology [3], where the rays of P constitute a subbase for the closed sets.

THEOREM 1. *Let S be a convex subset of a partially ordered set P, and let S be compact with respect to the Frink interval topology on P. Then S is the convex hull of its set of extreme points.*

Proof. It is sufficient to show that for each $x\in S$ there exist elements s minimal in S and t maximal in S, such that $s\leqq x\leqq t$. Let $x\in S$ be given. We shall show the existence of t; the existence of s is dual.

Let C be a maximal chain in $J(x)\cap S$, and let $M=\bigcap_{c\in C} J(c)$. Since each of the nested closed sets $J(c)$ has a nonempty intersection with the compact set S, the set $M\cap S$ is likewise nonempty. Let $t\in M\cap S$. Then $t\geqq x$. The fact that $c\leqq t$ for all $c\in C$ and the maximality of C together imply that t is a maximal element of C and hence a maximal element of S, as required. The proof is thus complete.