High School Mathematics Field Day Exam

Thursday, March 19, 2009

Southern Illinois University Carbondale

PLEASE READ THESE DIRECTIONS CAREFULLY!

1. Calculators, cell phones, iPods or any other electronic devices ***ARE NOT*** permitted. **Any calculator, cell phone, iPod or any other electronic device seen at any time after entering the auditorium will immediately disqualify that student.**

2. Use a pencil with No. 2 lead for marking the answer sheet. Only marks in the small circles on the answer sheet are recorded. Avoid smudges. Keep all pencil marks inside the little circles. Do not try to cover up the letter. If the letter is not fully inside the circle it is crucial that you mark the correct circle.

3. a) Follow oral instructions to enter your name.

   b) Enter your four-digit registration number. (This is the number on your 3 x 5 card.) Then blacken the corresponding numbers underneath.

   c) Failure to correctly fill in your code number is grounds for disqualification.

4. Indicate your answers on the answer sheet. No credit will be given for answers written in this examination booklet. Be sure that each mark on the answer sheet is black and **completely fills the circle.** Give only one answer to each question. No credit will be given for multiple answers. If you change an answer, be sure that all previous marks are completely erased. Avoid accidental marks in any other space.

5. Your score will be the number of correct answers marked minus one fourth of the number incorrect answers marked. This is to discourage guessing.

6. Do not spend too much time on any one question. Answer the easier questions first and then go back to the others if time permits.

7. Many questions, even those which look unfamiliar to you, require no specialized mathematical knowledge beyond 9th grade algebra. The questions are not in order of difficulty. If you get stuck, go on; there may be easier questions for you further on.

8. There are 2 blank pages at the end for scratch work.

9. **Begin the test only when told to do so.**

10. You may take the questions home with you.

11. Check the following carefully:
    a) Name is encoded according to the oral instructions.

    b) Your 4-digit registration number is coded. (This is the number on your 3 x 5 card.)

    c) Only one answer per question.

    d) There are 67 problems typed on 16 pages. The test is double-sided.

1. What is the value of $\left(1+\frac{2}{3}-\frac{3}{5}\right)\left(2+\frac{1}{4}\right)$?

a) $\frac{6}{5}$ b) $\frac{8}{5}$ c) 2 d) $\frac{12}{5}$ e) $\frac{14}{5}$

2. Simplify $\dfrac{\sqrt{x}\,y^{\frac{-1}{2}}\,\dfrac{\sqrt[4]{x^2y}}{y^2\sqrt{x+y}}}{\dfrac{x^{\frac{-1}{2}}\,y^{\frac{1}{4}}}{y\sqrt{xy^4+y^5}}}$.

a) $\sqrt{x^3y}$ b) $\sqrt{x^3y^5}$ c) $\sqrt{xy}$ d) $x\sqrt{x+y}$ e) No answer listed.

3. Consider the line $3x+2y=7$. The sum of the x and y intercepts is

a) $\frac{7}{2}$ b) $\frac{7}{3}$ c) $\frac{35}{6}$ d) $\frac{5}{7}$ e) $-\frac{5}{7}$

4. If $\log_a b+\log_b a=3,$ then what is the value of $(\log_a b)^2+(\log_b a)^2$?

a) 6 b) 7 c) 8 d) 9 e) 11

5. How many real solutions does the equation $\sqrt{x}=\left|x^3-1\right|$ have?

a) 0 b) 1 c) 2 d) 3 e) More than 3.

6. What is the area of a region in the first quadrant defined by $18 \le x + y \le 20$?

a) 36 b) 38 c) 40 d) 42 e) 44

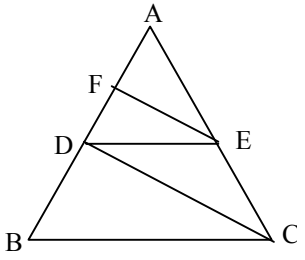
7. The sum $a + b$, the product ab, and the difference of squares $a^2 - b^2$ of two positive numbers a and b is the same nonzero number. What is b?

a) 1 b) $\frac{1+\sqrt{5}}{2}$ c) $\sqrt{3}$ d) $\frac{7-\sqrt{5}}{2}$ e) 8

8. If the width of a rectangle is increased by 10% and the length of the rectangle is increased by 20%, by what percentage does the area of the rectangle increase?

a) 2 % b) 10 % c) 15 % d) 32 % e) 36 %

9. In ΔABC, $DE \,//\, BC$, $EF \,//\, CD$, $AF = 4$, $FD = 6$. Find BD.



a) 5 b) 10 c) 15 d) 25 e) 30

10. The first and third terms of a geometric sequence are $a_1 = 4$ and $a_3 = 9$. What is the sixth term a_6?

a) 16 b) 24 c) 25 d) 12 e) $30\frac{3}{8}$

11. If x is required to be positive, what is the smallest possible value of $x + \frac{9}{x}$?

a) 2 b) 3 c) 4 d) 5 e) 6

12. Evaluate the following (without a calculator).

$$\tan 1° \cdot \tan 2° \cdot \tan 3° \cdot \ldots \cdot \tan 87° \cdot \tan 88° \cdot \tan 89°$$

a) 1 b) 0 c) $\tan 100°$ d) π e) 123,456,789

13. At night a man who is 6 ft. tall stands 5 ft. from a lamppost 16 ft. high. How long is his shadow?

a) $\frac{15}{8}$ ft. b) 3 ft. c) $\frac{10}{3}$ ft. d) 4 ft. e) 5 ft.

14. A fish swam 4 km downstream, turned around and swam 9 km back upstream, all during one hour. The speed of water is 2 *km/h*. What was the speed of the fish in standing water?

a) $\frac{9}{4}$ b) 5 c) 13 d) 14 e) 36

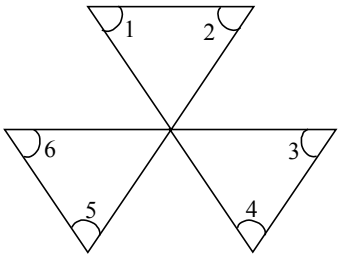
15. Let $x=(a+b)(c+d)$, $y=(a+c)(b+d)$ and $z=(a+d)(b+c)$. If $a<b<c<d$, then

a) $x<y<z$ b) $x<z<y$ c) $y<x<z$ d) $z<x<y$ e) No answer listed.

16. Solve for z: $z^2+2z=1$

a) $(1,0)$ b) $\frac{1\pm\sqrt{2}}{2}$ c) $\frac{\pm\sqrt{2}-1}{2}$ d) $-1\pm\sqrt{2}$ e) $\sqrt{2}\pm1$

17. Find the sum of the marked angles: $\angle1+\angle2+\Lambda\ +\angle6$



a) 180° b) 360° c) 450° d) 540° e) No answer listed.

18. Find the volume of a regular octahedron with side 1.

a) $\frac{\sqrt{2}}{12}$ b) $\frac{\sqrt{2}}{3}$ c) $\frac{\sqrt{2}}{6}$ d) $\frac{\sqrt{3}}{3}$ e) No answer listed.

19. Find the value of $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}$.

a) $-\frac{1}{2}$ b) $\frac{1}{2}$ c) $\frac{2\sqrt{3}}{5}$ d) $-\frac{2\sqrt{3}}{5}$ e) $\frac{\sqrt{3}+\sqrt{7}}{5}$

20. Numbers 1 and -2 are roots of the polynomial $2x^3+ax^2+bx+12$. The product, *ab,* is

a) -24 b) -14 c) 7 d) 40 e) 24

21. Solve for x: $\log_2(x+2)=\log_4(2x+12)$

a) -1 b) $\log_8 2$ c) 1 d) 2^{12} e) 2

22. An odd integer between 600 and 800 is divisible by both 7 and 9. What is the sum of its digits?

a) 7 b) 12 c) 18 d) 21 e) 27

23. Find: $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right)+\arccos\left(\cos\left(-\frac{2\pi}{3}\right)\right)$

a) 0 b) π c) $\frac{2\pi}{3}$ d) $\sqrt{\frac{12}{7}}$ e) No answer listed.

24. For what value of x do these three terms form an arithmetic sequence?

$$2x^2 - 3x, \quad x^2 + 2, \quad 6x + 1$$

a) -1 b) 0 c) 1 d) $1+\sqrt{2}$ e) 2

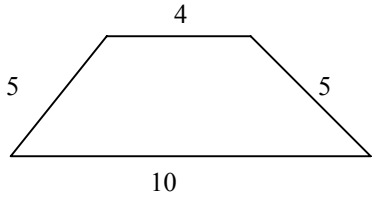
25. Two cards are drawn at random from a standard deck of 52 cards, without replacement. What is the probability of drawing a 7 and an ace in that order?

a) $\frac{4}{51}$ b) $\frac{4}{52}$ c) $\frac{7}{256}$ d) $\frac{4}{256}$ e) $\frac{4}{663}$

26. When $y = 3$, which of the following is false?

a) y is prime and y is odd b) y is odd or y is even c) y is not prime and y is odd d) y is odd and $2y$ is even e) $3y$ is not even

27. Find the area of the trapezoid.



a) 28 b) 7 c) 54 d) 14 e) Not enough information.

28. Last year, the population grew by 1% and the average income per person grew by 2%. By what percent did national income grow?

a) 1.05 % b) 3.21 % c) 2 % d) 3.02 % e) 1 %

29. A bakery owner turns on his doughnut machine at 8:30 a.m. At 11:10 a.m.. the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

a) 1:50 p.m. b) 3:00 p.m. c) 3:30 p.m. d) 4:30 p.m. e) 5:50 p.m.

30. Which of the following inequalities are true for all positive real numbers *a* and *b*?

i) $a^2 + b^2 \geq a + b$ iii) $a^2 + b^2 \geq ab$

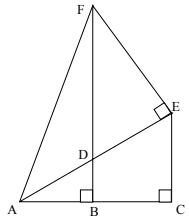
ii) $a^2 + b^2 \geq 2ab$ iv) $a^3 + b^3 \geq ab$

a) ii) and iii) only b) ii), iii), and iv) only c) ii) only d) iv) only e) i), ii), and iii) only

31. Suppose the points (x, y) in the first quadrant of the plane satisfy $x + y \leq 7$ and $3x + y \geq 12$. At what point does the maximum value of $6x + y$ occur?

a) $(4, 0)$ b) $(12, 0)$ c) $(7, 0)$ d) $(7, 12)$ e) No answer listed.

32. Given $AC = 7$, $CE = EF = 5$, compute BC.



a) $7 - \frac{25}{\sqrt{74}}$ b) $\frac{7\sqrt{74}+25}{74}$ c) $7 - \frac{49}{\sqrt{74}}$ d) $\frac{25}{\sqrt{74}}$ e) No answer listed.

33. At a father-daughter picnic, five men are seated in a circle with their five daughters. No man is seated next to another man or his own daughter. How many arrangements are possible?

a) 312 b) 13 c) 234 d) 432 e) No answer listed.

34. $\dfrac{\frac{a}{b}+\frac{3b}{a}}{\frac{a}{b}-\frac{b}{2a}} =$

a) $\frac{a+3b}{a-2b}$ b) $\frac{a^2+3b^2}{a^2-2b^2}$ c) $\frac{3a^2+b^2}{2a^2-b^2}$ d) $\frac{2a^2+6b^2}{2a^2-b^2}$ e) $\frac{3a+b}{2a-b}$

35. Let a and b be positive real numbers with $a + b = 4$. What is the minimum value of

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right) ?$$

a) 2 b) $\frac{8}{3}$ c) $\frac{9}{4}$ d) 3 e) 4

36. A square and an equilateral triangle have equal perimeters. The square has area 36 square units. What is the perpendicular height of the equilateral triangle?

a) $8\sqrt{3}$ b) $6\sqrt{3}$ c) $2\sqrt{3}$ d) $4\sqrt{3}$ e) No answer listed.

37. The number 86 in binary is

a) 1001100 b) 1100100 c) 1010010 d) 1010110 e) 1110000

38. What is the least number of cards which must be drawn from a deck of 52 cards to ensure that you have a 7 or a heart?

a) 38 b) 37 c) 36 d) 35 e) 34

39. Points *A* and *B* are on a circle of radius 5 and $AB = 6$. Point *C* is the midpoint of the minor arc *AB*. What is the length of the line segment *AC*?

a) $\sqrt{10}$ b) $\frac{7}{2}$ c) $\sqrt{14}$ d) $\sqrt{15}$ e) 4

40. For what value of parameter *m* does the following equation have exactly one solution?

$$x^2 + 10x + 21 = 2mx + 6m$$

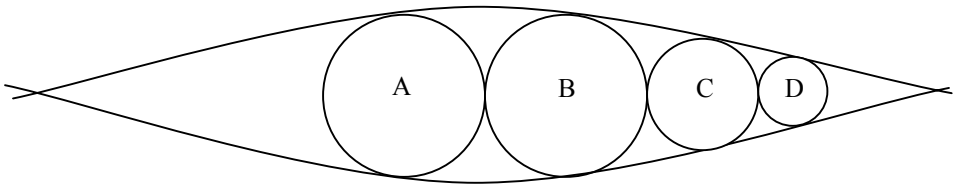
In which of these groups is your answer?

a) $0, \frac{1}{3}, \frac{1}{2}$ b) $\sqrt{2},\ \sqrt{3},\ 5$ c) $1,\ -7,\ 8$ d) $-1,\ 2,\ -\sqrt{2}$ e) $-3,\ 3,\ 6$

41. In how many ways can the numbers 1, 2, 3, 4, 5, 6, be arranged in a sequence *a, b, c, d, e, f* such that $a + f = b + e = c + d$?

a) 6 b) 24 c) 36 d) 48 e) 60

42. Circles *A* and *B* have radius 3, while *C* has radius 2. The arcs enclosing the four circles are pieces of two larger circles of equal radius. What is the radius of circle *D* ?



a) 4/3 b) 8/9 c) 1 d) 3/2 e) Not enough information

43. The numbers $\log\left(a^3 b^7\right)$, $\log\left(a^5 b^{12}\right)$, and $\log\left(a^8 b^{15}\right)$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log b^n$. What is *n*?

a) 40 b) 56 c) 76 d) 112 e) 143

44. Find the number of solutions to $x + y + z + w = 1$ in integers greater than -4.

a) Infinitely many. b) 560 c) 5985 d) 680 e) 1140

45. A cube has an edge that is 4 inches long. If the edge is increased by 25%, by what percentage is the volume increased? (Round your answer.)

a) 25% b) 56% c) 73% d) 95% e) 122%

46. Lilies planted in a lake double every night. Given the whole lake is covered in 30 days, in how many days will half the lake be covered?

a) 15 b) $\log 30$ c) 29 d) $\sqrt{30}$ e) 12

47. Let *P* be the point $(2, 2)$, *Q* be the reflection of *P* about the *x*-axis, *R* be the reflection of *P* about the line $y = -x$. Then *PQR* is a triangle. What is the area of *PQR*?

a) 10 b) 8 c) 4 d) 12 e) 16

48. How many four-digit numbers between 6000 and 7000 are there for which the thousands digits equal the sum of the other three digits?

a) 20 b) 22 c) 24 d) 26 e) 28

49. Consider the equation $\ln(2x+1) = \ln x + \ln(x+1)$ where $x > 0$. The equation has

a) no solution. b) 1 solution. c) 2 solutions. d) more than 2 solutions. e) infinitely many solutions.

50. An athlete covers three consecutive miles by swimming the first, running the second, and cycling the third. He runs twice as fast as he swims and cycles one and a half times as fast as he runs. He takes ten minutes longer than he would do if he cycled the whole three miles. How many minutes does he take?

a) 16 b) 22 c) 30 d) 46 e) 70

51. A quadrilateral $ABCD$ has vertices with coordinates $A(0,0)$, $B(6,0)$, $C(5,4)$, and $D(3,6)$. What is its area?

a) 18 b) 19 c) 20 d) 21 e) 22

52. A standard deck of 52 cards contains 13 hearts. Twenty-six cards have already been dealt eight of which are hearts. If you dealt 13 of the remaining cards, what is the probability that you will get exactly 2 of the 5 remaining hearts? (Round your answer.)

a) 22% b) 26% c) 30% d) 34% e) 38%

53. If we divide 344 by d the remainder is 3, and if we divide 715 by d the remainder is 2. Which of the following is true about d ?

a) $10 \le d \le 19$ b) $20 \le d \le 29$ c) $30 \le d \le 39$ d) $40 \le d \le 49$ e) $50 \le d \le 59$

54. Let α and β be two angles of a triangle Δ such that $\sin^2 \alpha + \sin^2 \beta = 1$. Which of the following statements is true?

a) Δ cannot be an acute triangle

b) Δ must be a right triangle

c) Δ must be an obtuse triangle

d) The difference between α and β cannot be more than $70°$.

e) None of the above statements is true.

55. Simplify $\sqrt{4+\sqrt{15}} - \sqrt{4-\sqrt{15}}$.

a) $\sqrt{10}$ b) $\sqrt{3}$ c) $\sqrt{6}$ d) $\sqrt{14}$ e) $2\sqrt{3}$

56. The perimeter of a rectangle is 34*cm*. Its diagonal is 13*cm*. Find the value of the sine of the angle determined by the diagonal and the longer side of the rectangle. Round to three decimal places.

a) 0.923 b) 0.385 c) $\frac{30}{34} \approx 0.882$ d) Exactly 1 e) .417

57. Let *s* be the set of all positive real numbers *x* satisfying the equation $2008^x = 2007^{2x}$. Then the number of elements of *s* is

a) 0 b) 1 c) 2 d) a finite number greater than 2. e) $+\infty$

58. What single discount is equal to two successive discounts of 10% and 15%?

a) 25% B) 24.5% c) 24% d) 23.5% e) 22%

59. Let *ABCD* be a rectangle. Points *P, Q , R , S* are chosen on the sides *AB, BC, CD, DA,* respectively, such that $\frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RD} = \frac{DS}{SA} = \frac{2}{3}$. Then the ratio $\frac{\text{Area of } PQRS}{\text{Area of } ABCD}$ equals

a) $\frac{13}{25}$ b) $\frac{9}{25}$ c) $\frac{11}{25}$ d) $\frac{4}{13}$ e) $\frac{16}{97}$

60. What is the area of the region defined by the inequality $|2x-4|+|y+1| \le 2$?

a) 3 b) $\frac{7}{2}$ c) 4 d) $\frac{9}{2}$ e) 5

61. $\sin^2 x \cos^2 x =$

a) $\frac{1}{8} - \frac{1}{8}\cos 4x$ b) $\frac{1}{2} - \frac{1}{2}\cos 2x$ c) $\frac{1}{8} - \frac{1}{8}\cos 2x$ d) $\sin 2x \cos 2x$ e) No answer listed.

62. For an angle α in the first quadrant, it is known that $\frac{\cos 3\alpha}{\cos \alpha} = \frac{1}{3}$. Find $\frac{\sin 3\alpha}{\sin \alpha}$.

a) $\frac{7}{3}$ b) $\frac{3}{7}$ c) $\frac{4}{7}$ d) $\frac{\sqrt{7}}{3}$ e) $\frac{7}{\sqrt{3}}$

63. Given two parallel lines of distance 1 apart and a circle of radius 2. The circle is tangent to one of the lines and cuts the other line. The area of the circular cap between the two parallel is $\frac{a\pi}{3} - b\sqrt{3}$. Find the sum $a + b$ of the two integers *a* and *b*.

a) 3 b) 4 c) 5 d) 6 e) 7

64. If 1200cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

a) 4000 b) 3000 c) 3750 d) 4500 e) No answer listed.

65. Let the decimal number *x* have base 2 representation 10101. Let the decimal number *y* have base 3 representation 10101. What is the base 5 representation of the decimal number *xy?*

a) 31021 b) 20303 c) 10101 d) 30121 e) 10321

66. If you know that $\sin\alpha > 0$ and that $3\tan\alpha = 6\sin^2\alpha$, find $\cot\alpha$.

a) 0.6 b) 15 c) 1 d) .75 e) 12

67. Two distinct prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers can be obtained?

a) 21 b) 60 c) 119 d) 180 e) 231