Applications of Chromatic Polynomials Involving Stirling Numbers

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The Stirling numbers of the second kind, denoted $S(n,k)$, are the number of ways to partition n distinct objects into k nonempty subsets. We use the notation $[n] = \{1, 2, \ldots, n\}$ and sometimes refer to the subsets as blocks. The initial conditions are defined as: $S(0,0) = 1$, $S(n,0) = 0$, for $n \geq 1$, and $S(n,k) = 0$ for $k > n$. We also have $S(n,2) = 2^{n-1} - 1$ and $S(n, n-1) = \binom{n}{2}$.

Example 1. $S(4,2) = 7$

We represent a given partition, e.g., $\big\{\{1,3\},\ \{2,4\}\big\}$ in block notation as $1,3|2,4$. So the 7 blocks are:

$$1|2,3,4 \quad 2|1,3,4 \quad 3|1,2,4 \quad 4|1,2,3 \quad 1,2|3,4 \quad 1,3|2,4 \quad 1,4|2,3$$

The numbers $S(n,k)$ satisfy the following well-known triangular recurrence. We go ahead and include a common proof argument as well. See, for example, [1].

Theorem 1. *$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$ for all positive integers n, k, and $n \geq k$.*

Proof. The set of partitions of $[n]$ into k subsets can be partitioned into two disjoint cases. Case A contains all partitions where the object 1 is in a block by itself. There are $S(n-1,k-1)$ such partitions. Case B contains all partitions where the object 1 is not by itself. It follows that there are $kS(n-1,k)$ such partitions. □

An application of Stirling numbers is that $k!S(n,k)$ equals the number of surjections from an n-set onto a k-set. This follows since any surjection $f:[n]\to[k]$ naturally induces a partition of $[n]$ into k-nonempty sets.

We now give a generalization of the numbers $S(n,k)$.

Definition 1. For a positive integer d, let $S^d(n,k)$ denote the number of partitions of $[n]$ into exactly k nonempty subsets such that for each subset, all elements in the subset have pairwise distance at least d. So, for any i,j in a given subset we require $|i-j|\geq d$. Notice the case $d=1$ yields the classical Stirling numbers of the second kind, $S(n,k)$.

In general, for the numbers $S^d(n,k)$, $d,n,k\geq 1$, we define the following initial conditions: $S^d(1,1)=1$, $S^d(n,1)=0$ for $n\geq 2$, and $S^d(n,k)=0$ for $k>n$. We first focus on the numbers $S^2(n,k)$. Identities we obtain when $d=2$, in conjunction with chromatic polynomials on the path will yield alternative methods to derive some well-known Stirling number identities. The case when d= 2 is given as an exercise in the classic book by D. Cohen [2] in the context of a banner coloring problem. The main result in this paper is an explicit formula for $S^d(n,k)$ where d is any positive integer.

Example 2. $S^2(5,3)=7$.
We list the 7 blocks:

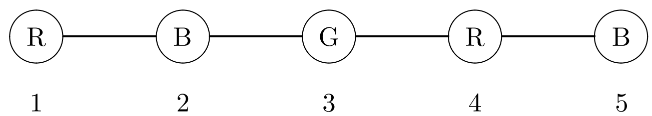
$$1,3|2,4|5\ ;\quad 1,3|2,5|4\ ;\quad \boxed{1,4|2,5|3}$$

$$1,4|2|3,5\ ;\quad 1,4|3,5|2\ ;\quad 1,5|2,4|3\ ;\quad 1,3,5|2|4$$

We now introduce the *inclusive* chromatic polynomial of a graph. For any basic graph terminology, consult [3]. A *proper* vertex-coloring of a graph G is a coloring of its vertex set so that adjacent vertices are assigned different colors. The chromatic polynomial of a graph G, denoted $\chi(G,k)$ is the number of proper vertex colorings of G that use k (or fewer colors).

The *inclusive* chromatic polynomial of G, denoted $\tilde{\chi}(G,k)$ is the number of proper colorings of $V(G)$ that use *exactly* k-colors. Let P_n denote the path on n-vertices.

Example 3. $\tilde{\chi}(P_5, 3) = 3! \cdot 7 = 42$.
This example is related to Example 2. Consider the properly colored path P_5 using exactly 3-colors (red (R), blue (B), green (G)).



Notice this coloring concerns the circled block in Example 2, namely $1, 4|2, 5|3$. Here the block $\{1, 4\}$ is colored red, $\{2, 5\}$ blue and $\{3\}$ green. Clearly each block can be colored in any 3! ways of the permutings of the colors. Also corresponding to the definition of $S^2(5, 3)$, each partition induces a proper coloring using three colors.

Theorem 2. $\tilde{\chi}(P_n, k) = k!S^2(n, k)$.

Proof. Consider the set, $\mathcal{F}$, of all partitions given by the definition of $S^2(n, k)$. With $V(P_n) = [n]$, we have that any partition yields $k!$ proper colorings of P_n using exactly k colors. Conversely, consider any proper coloring using the k-colors, $c_1, c_2, \ldots, c_k$, Let Q_i denote the set of vertices colored c_i . Then $Q_1, Q_2, \ldots, Q_k$ is in $\mathcal{F}$. Also, the blocks in any partition of $\mathcal{F}$ are not labelled, so if we interpret a permutation of $Q_1, Q_2, \ldots, Q_k$ to mean that if Q_i is now in the jth position, then the vertices in Q_i are now colored c_j . So, permuting $Q_1, \ldots, Q_k$ generates $k!$ proper colorings. □

For a general graph G, we can obtain a formula for $\tilde{\chi}(G, k)$ by the principle of inclusion and exclusion. Let A_i denote the set of proper colorings that do not use color i. We then have that $\tilde{\chi}(G, k) = |\overline{A}_1 \cap \overline{A}_2 \cdots \cap \overline{A}_k|$. Notice, for example, $|A_1| = \chi(G, k-1)$ and $|A_1 \cap A_2| = \chi(G, k-2)$. By the inclusion/exclusion formula we have:

Theorem 3. *Let G be a connected graph of order $n \geq 2$ and $k \leq n$.*

$$\tilde{\chi}(G, k) = \chi(G, k) - \binom{k}{1}\chi(G, k-1) + \binom{k}{2}\chi(G, k-2) \pm \cdots \binom{k}{k-2}\chi(G, 2).$$

□

Notice we omit the usual last two terms on the right-hand side of Theorem 3, since with $n \geq 2$, $\chi(G, 1) = \chi(G, 0) = 0$. Using the well-known fact that $\chi(P_n, k) = k(k-1)^{n-1}$ and Theorems 2 and 3 yields the following explicit formula for $S^2(n, k)$.

Theorem 4. *For $n \geq 2$ and $k \leq n$ we have,*

$$S^2(n,k) = \frac{1}{k!}\tilde{\chi}(P_n,k) = \frac{1}{k!}\sum_{i=0}^{k-2}(-1)^i\binom{k}{i}(k-i)(k-i-1)^{n-1}$$

□

To illustrate Theorem 4 we have:

Example 4.

$$\begin{aligned} S^2(5,3) &= \frac{1}{3!}\tilde{\chi}(P_5,3) = 7 \qquad\qquad \text{(see ex. 3)} \\ &= \frac{1}{6}\left(3\cdot 2^4 - \binom{3}{1}(2)(1)^4\right) = \frac{1}{6}(48-6). \end{aligned}$$

□

We now show $S^2(n,k) = S(n-1,k-1)$. We first need the following triangular recurrence for $S^2(n,k)$.

Theorem 5. *For $n,k \geq 2$,*

$$S^2(n,k) = S^2(n-1,k-1) + (k-1)S^2(n-1,k).$$

Proof. By the definition of $S^2(n,k)$, the set of partitions of $[n]$ into k subsets can be partitioned into two disjoint cases. Case A contains all partitions where the object 1 is in a block by itself. There are $S^2(n-1,k-1)$ such partitions. Case B contains all partitions where 1 is not by itself in a block. It follows that we can take any of the $S^2(n-1,k)$ partitions of the set $\{2,\ldots,n\}$ into k bocks, where each block contains integers that are pairwise distance at least two. Now, we can insert 1 into any block not containing 2. So we have that there are $(k-1)S^2(n-1,k)$ partitions for Case B. □

We have the following theorem, which is why we call the numbers $S^2(n,k)$ *reduced* Stirling numbers.

Theorem 6. $S^2(n,k) = S(n-1,k-1)$ *for $n,k \geq 2$.*

Proof. The proof is by induction on $n+k$. For the ground case $(n+k)=4$ and $n,k \geq 2$ we have $(n,k)=(2,2)$ and $S^2(2,2) = S(2-1,2-1) = 1$

holds. Assume $S^2(n,k) = S(n-1,k-1)$ for all n, k where $4 \le n+k \le m$ and let $n+k = m+1$. Then

$$\begin{aligned} S^2(n,k) &= S^2(n-1,k-1) + (k-1)S^2(n-1,k) && \text{(by Thm. 5)} \\ &= S(n-2,k-2) + (k-1)S(n-2,k-1) && \text{(by ind. hyp.)} \\ &= S(n-1,k-1). && \text{(by Thm. 1).} \end{aligned}$$

□

We remark that we used a version of the method of mathematical induction here to prove two arrays of numbers are equal. Namely we first verified they have the same initial conditions, then showed they satisfied the same recurrence relation.

Combining Theorems 4 and 6 we obtain an explicit formula for the inclusive chromatic polynomial of the path, namely:

$$S^2(n,k) = S(n-1,k-1) = \frac{1}{k!}\tilde{\chi}(P_n,k). \tag{1}$$

We can also replce $\tilde{\chi}(P_n,k)$ with the formula given in Theorem 3, e.g., with $n = 7$ and $k = 4$ we have:

$$S^2(7,4) = S(6,3) = 90 = \frac{1}{4!}(2916 - 768 + 12).$$

Notice we can combine the formulas in Theorems 3, 4, and 6, with a change of variables we obtain the well-known formula

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n. \tag{2}$$

The usual method to obtain (2) is by the exponential generataing function for $S(n,k)$. We can obtain another well-known identity as follows: Notice

$$\chi(G,k) = \sum_{j=1}^{k}\binom{k}{j}\tilde{\chi}(G,j), \tag{3}$$

since the number of ways to color G with k (or fewer) colors is obtained by the RHS of (3). Setting $G = P_n$, with $n \ge 2$ we then have:

$$k(k-1)^{n-1} = \sum_{j=2}^{k}\binom{k}{j}j!S(n-1,j-1). \tag{4}$$

It is straightforward to see (4) is after a change of variables, equivalent to the well-known identity

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \tag{5}$$

where $(x)_k = x(x-1)\cdots(x-k+1)$. As an example using (4) with $n=5$, $k=3$ we obtain

$$\begin{aligned} 3(16) &= \binom{3}{2} 2! S(4,1) + \binom{3}{3} 3! S(4,2) \\ &= 6 + 6(7) = 48. \end{aligned}$$

Another observation that in turn can generate identities, is that if G is of order n and $k > n$, then clearly $\tilde{\chi}(G,k) = 0$. If again we let $G = P_n$, and $k > n$, by Theorem 3 we obtain

$$\sum_{i=0}^{k-2} (-1)^i \binom{k}{i} \chi(P_n, k-i) = 0.$$

For example, with $k=4$ and $n=3$ we obtain

$$4(3)^2 - \binom{4}{1} 3 \cdot 2^2 + 6(2) = 0.$$

We now consider the general case of $S^d(n,k)$ where again, $S^d(n,k)$ is the number of partitions of $[n]$ into k subsets such that for each subset, all elements in the subset have pairwise distance at least d. Also $S^d(d,d) = 1$ and $S^d(n,k) = 0$ for $n < k$.

Example 5. $S^3(6,4) = 7$.

The seven partitions are:

$$1,4|2,5|3|6; \quad 1,4|2|5|3,6; \quad 1,4|2,6|3|5$$

$$1,5|2,6|3|4; \quad 1,5|2|3,6|4; \quad 1,6|2,5|3|4; \quad 1|2,5|3,6|4$$

We have the following triangular recurrence for $S^d(n,k)$.

Theorem 7. *$S^d(n,k) = S^d(n-1,k-1) + (k-d+1)S^d(n-1,k)$, $n \geq k \geq d$.*

Proof. If we split the set $\{1,\ldots,n\}$ into k subsets, either there is a subset $\{n\}$ or not. The number of ways in which $\{n\}$ is one of the subsets is $S^d(n-1,k-1)$ since we must split the remainder $\{1,\ldots,n-1\}$ into the other

$k-1$ subsets. If n belongs to a subset with other elements, we can delete it and obtain a partition of the set $\{1, \ldots, n-1\}$ into k subsets, of which there are $S^d(n-1,k)$. Now, we know that n cannot be reinserted into a subset containing any of $n-1, n-2, \ldots, n-d+1$, each of which is guaranteed to be in a unique subset since the distance between any two of them is less than d. Hence, we have a total of $\left[k - \left[(n-1)-(n-d+1)+1\right]\right] * S^d(n-1,k) = (k-d+1) * S^d(n-1,k)$ choices for reinserting n. Adding these two totals yields the result. □

The proof of the following theorem is analogous to that presented in Theorem 6.

Theorem 8. $S^d(n,k) = S(n-d+1, k-d+1)$ *where* $n \geq k \geq d$.

Proof. The proof is by induction on $n+k$. We have the ground case where $S^d(d,d) = S(1,1) = 1$. Now assume the formula is true for all $n+k$, where $2d \leq n+k \leq m$. Now consider $n+k = m+1$. We have, by Theorem 7 that

$$\begin{aligned} S^d(n,k) &= S^d(n-1,k-1) + (k-d+1)S^d(n-1,k) \\ &= S(n-d,k-d) + (k-d+1)S(n-d,k-d+1) \\ &= S(n-d+1,k-d+1). \end{aligned}$$

The last equality is from Theorem 1. □

We observe that Theorem 8 yields the following formulas:

$$S^d(n,k) = S^{d-1}(n-1,k-1) \tag{6}$$

$$S(n,k) = S^{d+1}(n+d,k+d), \quad \text{for } d \geq 1. \tag{7}$$

Equations (6) and (7) are illustrated in the triangular table of $S(n,k)$ (see Fig 1). For example, we have the following equalities:

$$15 = S(5,2) = S^2(6,3) = S^3(7,4) = \cdots$$

We may continue down the array and obtain, for example, that $S^{2003}(2007, 2004) = 15$. So the number of partitions of [2007] into 2004 blocks, where for any i, j in a block, $|i-j| \geq 2003$ is $15 = S(5,2)$.

We circled the number 15 in Fig 1 to represent it as a *seed*, and as we travel down the diagonal the corresponding reduced Stirling numbers are a constant by Eq. (7). We leave as an open problem to explicitly give the bijection between sets of partitions along their diagonal from a given seed.

Stirling Numbers of the Second Kind
n from 1 to 10

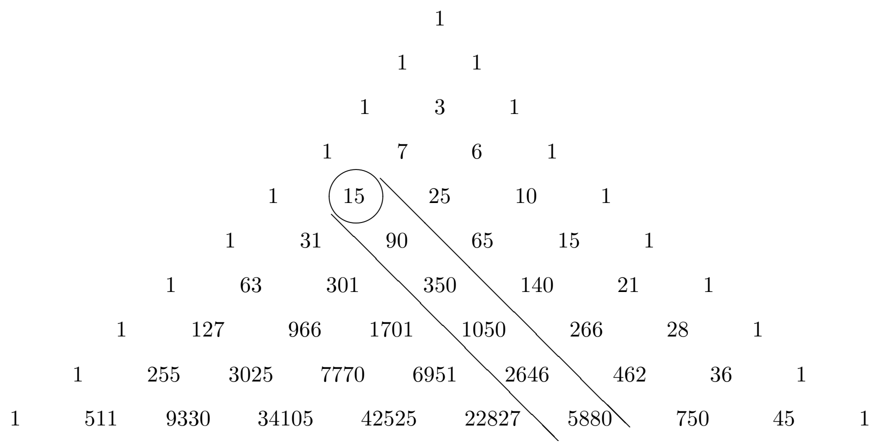


Figure 1:

References

[1] M. Bona, *Introduction to Enumerative Combinatorics*, McGraw Hill, 2007.

[2] D.I.A. Cohen, *Basic Techniques of Combinatorial Theory*, John Wiley & Sons, 1978.

[3] D. West, *Introduction to Graphy Theory*, Prentice Hall, 2001.