Characterizing Balanced Bipartite Graphs With Part-Switching Automorphisms

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Abstract

A balanced bipartite graph may have an automorphism that maps the vertices of one part to vertices of the other part. Such graphs have a symmetric bipartite adjacency matrix. We show that such graphs are precisely those graphs that may be factored as a tensor product of some graph with a complete graph on two vertices. We further present a construction to show that regular bipartite graphs without this property exist for all even $n \geq 6$ and degree 3.

The *tensor product* $G \otimes H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set consisting of those pairs of vertices $(g, h), (g', h')$ where g is adjacent to g' and h is adjacent to h'.

This product, also called the Kronecker product, weak product, direct product, categorical product, and conjunction, has been studied for decades. Basic properties of the tensor product may be found, for example, in [5, 3, 4]. Despite this study, many basic properties of the product are unknown or only partly understood. In this paper we characterize a certain kind of balanced bipartite graph in terms of this product. Notice that $K_2 \otimes G$ is a balanced bipartite graph for any G. For a bipartite graph G with parts X and Y, the *bipartite adjacency matrix* $A(G)$ has rows indexed by elements of X, and columns indexed by elements of Y. $A(G)$ has a one in position (i, j) if i is adjacent to j and a zero otherwise. $A(G)$ is *symmetric* if its (i, j)-element is equal to its (j, i)-element. $K_2 \otimes G$ need not be regular. We will use the convention that a loop at a vertex adds only one to the degree of the vertex.

Our main result shows the existence of regular bipartite graphs $G = (X, Y)$ where, under any permutation of X or Y, $A(G)$ cannot be made symmetric. We use the usual definition of an involution θ, i.e., θ is in the symmetric group S_n with $\theta^2 = (1)$.

Theorem 1: *The following conditions on a bipartite graph H are equivalent:*

(1) H has a symmetric bipartite adjacency matrix with constant row-sum.

(2) H may be written as $K_2 \otimes G$ for some regular G, where G may have loops.

(3) H is regular and has an involution that maps vertices in one part of H to vertices in the other part.

Proof: Suppose H has the required matrix. The ith row of $A(H)$ corresponds to a vertex of X which we will label (a, i); the ith column of $A(H)$ corresponds to a vertex of Y which we label (b, i). By the symmetry of $A(H)$, there is an edge in H from (a, i) to (b, j) if and only if there is an edge from (a, j) to (b, i). Thus $A(H)$ has constant column sum equal to its constant row-sum; call this common sum d. Since $A(H)$ is symmetric it must be square; let n be the number of rows of $A(H)$. Define a graph G on the vertices $\{1, 2, \ldots n\}$ where i is adjacent to j in G if and only if (a, i) is adjacent to (b, j) in H. It is now easy to confirm that $K_2 \otimes G$ is isomorphic to H, and the degree of each vertex G is d; thus (1) $\Rightarrow$ (2). Now (2) $\Rightarrow$ (3) is immediate since the involution corresponds to the matrix symmetry. Finally, (3) $\Rightarrow$ (1) follows from requiring that the vertex corresponding to column i be the image under the given involution of the vertex corresponding to row i, for $1 \le i \le n$. □

A slightly weaker result holds for non-regular balanced bipartite graphs through a minor modification of the proof.

Corollary 2: *The following conditions on a bipartite graph H are equivalent:*

(1) H has a symmetric bipartite adjacency matrix.

(2) H may be written as $K_2 \otimes G$ for some G, perhaps with loops.

(3) H has an involution that maps vertices in one part of H to vertices in the other part.

The proof runs just as above, neglecting the regularity and constant row-sum conditions.

There are regular bipartite graphs for which the three conditions of Theorem 1 are false. A smallest counterexample is shown in Figure 1. Notice that the vertices $5'$ and $6'$ have the same neighborhoods, but no two vertices of the other part have the same neighborhoods. Thus no automorphism can exchange the two parts. In general, we may construct a bipartite 3-regular graph on $2n + 6$ vertices of this type with the following construction. Let the two parts be $A = \{a_1, \ldots, a_{n+3}\}$ and $B = \{b_1, \ldots, b_{n+3}\}$. Then place edges between b_i and a_i, a_{i+1}, and a_{i+2}, for $1 \le i \le n - 1$; and edges from b_n to a_1, a_n, and a_{n+2}; edges from b_{n+1} to a_1, a_2, and a_{n+3}; edges from

b_{n+2} to a_{n+1}, a_{n+2}, and a_{n+3}; and edges from b_{n+3} to a_{n+1}, a_{n+2}, and a_{n+3}.

Figure 1

How many balanced bipartite graphs are there with this symmetry? This question can be approached by means of Theorem 1 and Corollary 2 in the case of labelled graphs. For it is not difficult to see that a graph on n vertices with loops possible may have as many as $\binom{n+1}{2}$ edges (as we disallow multiple edges, no vertex may have two loops). Hence there are $2^{n(n+1)/2}$ labelled balanced bipartite graphs on $2n$ vertices with the part-switching involution. Since there are $2^{n!}$ labelled balanced bipartite graphs in $2n$ vertices altogether, we see that without the assumption of regularity this is a relatively rare class of graph. We can also see that the problem of counting unlabelled graphs of this sort is exactly as difficult as counting unlabelled graphs in general.

Finally, we comment that many regular bipartite graphs may be described as a tensor product of the specified type. For example, the case of the n-dimensional cube Q_n, explicitly handled in [2], follows since $Q_n \cong K_2 \otimes Q^*_{n-1}$ where Q^*_{n-1} is the $(n-1)$-dimensional cube with a loop at each vertex.

References

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