Graphs with the Anti-Neighborhood-Sperner Property

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Abstract

In this note we introduce the concept of anti-neighborhood Sperner graphs. We give various conditions and constructions.

1 Introduction

In this paper the graphs are loopless and without multiple edges. The notation and terminology follow that used in [4]. A set system $\mathcal{F}$ is called a *Sperner family* if no member of the family is a subset of another. Another term for Sperner families is antichain. The sets under consideration are the open vertex neighborhoods of a graph G i.e., with $V(G) = \{1, 2, \ldots, n\}$ then $\mathcal{F} = \{N(1), N(2), \ldots, N(n)\}$. We may view $\mathcal{F}$ as the set of edges of a hypergraph. Let $H = \{E_1, E_2, \ldots, E_m\}$ be a simple hypergraph. We say that H has the *Helly* property if every intersecting family of H is a star, i.e. for $J \subset \{1, 2, \ldots, m\}$, $E_j \cap E_k \neq \emptyset$ for all $(j, k \in J)$ implies $\bigcap\limits_{j \in J} E_j \neq \emptyset$. In [1], [2], [3], the authors find applications for this set if $\mathcal{F}$ is both Helly and Sperner.

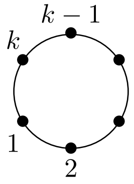
Here, the authors considered the family $\mathcal{F} = \{N(1), \ldots, N(n)\}$ from a given graph G. If $\mathcal{F}$ is both Helly and Sperner then they made operations on

G, in particular took its strict square, and constructed *self-clique* graphs. The clique graph of a graph G is the intersection graph, $K(G)$, of the (maximal) cliques of G. A graph G is called *self-clique* whenever $G \cong K(G)$. In this note we consider the same family $\mathcal{F}$, but we require it to be *anti-Sperner*. A set system $\mathcal{F}$ is called *anti-Sperner* if every member of the family is a subset of some other. For a given graph G, if its open neighborhood family $\mathcal{F} = \{N(1), \ldots, N(n)\}$ is anti-Sperner we say G is an *anti-neighborhood Sperner (ANS) graph*. For example, C_4 is ANS, also $K_{3,3} - e$ is ANS. We list some properties of ANS graphs.

2 Properties

Theorem 2.1. *If G is an ANS graph then the girth of G is less than 5.*

Proof. By contradiction, suppose G is ANS with girth ≥ 5. Let

$C =$ 

be a smallest cycle in G of length $k \geq 5$. Notice $N(1)$ is not contained in $N(i)$ for any $i \in \{2, \ldots, k\}$. Hence there must exist some vertex u not in $\{1, 2, \ldots, k\}$ with $\{2, k\} \subset N(u)$. But then $u, 2, 1, k, u$ is a 4-cycle, contradicting G has girth ≥ 5. $\square$

Theorem 2.2. *If G is a finite ANS graph then there must exist two vertices x and y with $N(x) = N(y)$.*

Proof. Let x be a vertex of an ANS graph G such that $|N(x)| \geq |N(u)|$ for all other vertices u. Since G is ANS there exists a vertex y with $N(x) \subseteq N(y)$, hence $|N(x)| \leq |N(y)|$. But by definition of x, $|N(x)| \geq |N(y)|$, hence $N(x) = N(y)$. $\square$

3 Constructions

We now list some constructions of graphs that are ANS. We are primarily interested in infinite graphs.

Proposition 3.1. *For finite graphs the complete multipartite graph* $K_{a_1,a_2,\ldots,a_n}$ *with* $|a_i| \geq 2$ *for* $i \in [n]$ *is ANS.*

Various selected subgraphs of this graph are also ANS. A more interesting iterative construction uses the tensor product as follows. The *tensor product* of two graphs G and H, denoted $G \otimes H$ is defined as $V(G \otimes H) = V(G) \times V(H)$ and $(g,h)(g',h')$ is an edge in $G \otimes H$ iff $gg' \in E(G)$ and $hh' \in E(H)$.

Theorem 3.2. *If* G *and* H *are ANS graphs, then* $G \otimes H$ *is an ANS graph.*

Proof. Let $u = (g,h)$ be a vertex in $G \otimes H$. Since G and H are ANS graphs there exist vertices g',h' in $V(G), V(H)$ such that $N_G(g) \subseteq N_G(g')$ and $N_H(h) \subseteq N_H(h')$. Let $v = (g',h')$. We show $N_{G\otimes H}(u) \subseteq N_{G\otimes H}(v)$. Let $w = (w_1,w_2) \in N_{G\otimes H}(u)$, then $w_1 \in N_G(g) \subseteq N_G(g')$ and $w_2 \in N_H(h) \subseteq N_H(h')$ hence $(w_1,w_2) \in N_{G\otimes H}(v)$. □

We would like to find examples of r-regular infinite ANS graphs. Below is an example of a 4-regular infinite ANS graph. It is a modification of the infinite ladder graph.

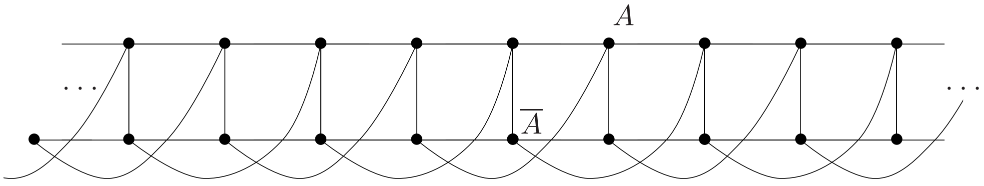


Figure 1: M_∞

We denote this graph by M_∞. The ANS property is illustrated by the labeled pairs $A, \overline{A}$, where $N(A) = N(\overline{A})$.

Theorem 3.3. *For any $r = 4q$ there exists an infinite r-regular ANS graph.*

Proof. For the case $q = 1$, the graph M_∞ is a candidate. For $q \geq 2$ we have by Prop. 3.1, that $K_{q,q}$ is a q-regular ANS graph. By Theorem 3.2, $K_{q,q} \otimes M_\infty$ is an ANS graph, it is also $4q$-regular. $\square$

In this short note we have introduced the topic of ANS graphs. From Theorem 3.3 we have the existence of r-regular infinite ANS graphs for $r \equiv 0 \mod 4$. We would like to find other examples for any r. We would also like to find more sufficient and necessary conditions for finite ANS graphs. Finally, we pose the question of the algorithmic complexity to recognize ANS graphs.

References

[1] F. Larrión, v. Neumann-Lara, M.A. Pizaña and T.D. Porter. *A hierarchy of self-clique graphs*. Submitted.

[2] F. Larrión, v. Neumann-Lara, M.A. Pizaña and T.D. Porter. *Self-clique graphs with prescribed clique-sizes*. Congresus Numerantium **157** (2002), pp. 173–182.

[3] F. Larrión, v. Neumann-Lara, M.A. Pizaña and T.D. Porter. *Recognizing self-clique graphs*, Mathematica Contemporânea **25** (2003), pp. 125–133.

[4] D.B. West, Introduction to Graph Theory, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2001.