

Binomial Identities Generated by Counting Spanning Trees

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Abstract

We partition the set of spanning trees contained in the complete graph K_n into spanning trees contained in the complete bipartite graph $K_{s,t}$. This relation will show that any property of spanning trees in K_n can be derived from trees in $K_{s,t}$. We enumerate the trees in K_n and $K_{s,t}$ recursively, and after applying the inclusion/exclusion principle of counting, we obtain some combinatorial and numerical identities. Among them are identities for n^p , where n and p are integers.

Keywords. Cayley's Theorem, spanning trees, binomial identities

1 Introduction

We use the standard notation and terminology which can be found, e.g., in [9]. Let $\tau(G)$ denote the number of labelled spanning trees in a graph G . With K_n denoting the complete graph of n vertices and $K_{s,t}$ the complete bipartite graph with partite sets containing s , respectively, t vertices. It is well known [1, 2, 4, 5, 6, 7]:

$$\tau(K_n) = n^{n-2}, \quad n \geq 2 \quad (1)$$

$$\tau(K_{s,t}) = s^{t-1}t^{s-1}, \quad s, t \geq 1. \quad (2)$$

We remark that (1) is often referred to as Cayley's theorem. Let $s + t = n$, where $1 \leq s \leq t$. We have the following observation:

Theorem 1.1. *With $n \geq 2$, any spanning tree T in K_n , is a spanning tree in $K_{s,t}$ for a unique pair (s, t) where $1 \leq s \leq t$ and $s + t = n$.*

Proof. Consider any spanning tree T in K_n , then T is a connected bipartite graph and as is well known, necessarily possesses a unique bipartition. So construct this unique bipartition by properly 2-coloring the vertex set of T with colors red (R) and blue (B). Let the number of red vertices be s and the number of blue vertices be t , w.l.o.g. let $s \leq t$. We then have $T \subset K_{s,t}$. \square

The converse is straightforward.

Theorem 1.2. *With $s+t = n$ any spanning tree in $K_{s,t}$ is a spanning tree in K_n .*

Proof. This follows since $K_{s,t} \subset K_n$. \square

Theorem 1.3. $\tau(K_n) = \sum_{s=1}^{\lfloor n/2 \rfloor} \binom{n}{s} \tau(K_{s,n-s})$.

Proof. By combining Theorems 1.1 and 1.2 we see that to find $\tau(K_n)$ we can enumerate all labelled spanning trees in the possible $K_{s,t}$ graphs. With $1 \leq s \leq t$ and $s+t = n$, this is the RHS summation. \square

We rewrite Theorem 1.3 as:

$$\sum_{s=1}^{n-1} \binom{n}{s} \tau(K_{s,n-s}) = 2\tau(K_n). \quad (3)$$

Substituting equations (1) and (2) into (3) yields the identity:

$$\sum_{s=1}^{n-1} \binom{n}{s} s^{n-s-1} (n-s)^{s-1} = 2n^{n-2}. \quad (4)$$

An analytic proof of (4) is forthcoming [3], where we derive the RHS of (4) from the LHS by using Abel's binomial formula. So in this sense, knowledge of $\tau(K_{s,t})$ implies $\tau(K_n)$, yielding a new proof of Cayley's theorem. The ideas in Theorems 1.1, 1.2 are also valid when graphs are unlabelled, since the unique bipartition aspect is a structural property of the graph G . So, for a connected graph G , let $I(G)$, be the number of non-isomorphic spanning trees in G , we have:

Theorem 1.4. $I(K_n) = \sum_{s=1}^{\lfloor n/2 \rfloor} I(K_{s,n-s})$. \square

Observational examples of Theorem 1.4 are:

$$\begin{aligned} I(K_6) &= 6 = I(K_{1,5}) + I(K_{2,4}) + I(K_{3,3}) \\ &= 1 + 2 + 3 \end{aligned}$$

$$\begin{aligned} I(K_7) &= 11 = I(K_{1,6}) + I(K_{2,3}) + I(K_{3,4}) \\ &= 1 + 3 + 7. \end{aligned}$$

Theorem 1.4 suggests a different approach to enumerating $I(K_n)$ from the usual approach done by Polya and Otter. Since a general formula of $I(K_{s,t})$ is unknown, it motivates us to research $I(K_{s,t})$ more deeply.

Getting back to equation (4), one can find a similar formula on Prof. László Székely's home web-page [8], where there is an informative ps, pdf file on Abel's binomial theorem

$$\text{(Székely)} \quad \sum_{s=1}^{n-1} \binom{n}{s} s^{s-1} (n-s)^{n-s-1} = 2(n-1)n^{n-2}. \quad (5)$$

Equating (4) and (5) yields an interesting identity:

$$(n-1) \sum_{s=1}^{n-1} \binom{n}{s} s^{n-s-1} (n-s)^{s-1} = \sum_{s=1}^{n-1} \binom{n}{s} s^{s-1} (n-s)^{n-s-1}. \quad (6)$$

We now derive recursive formulas for $\tau(K_{s,t})$ and $\tau(K_n)$ that yield corresponding identities. For a graph G with vertex set $V(G) = \{1, 2, \dots, n\}$, let A_i denote the set of spanning trees T in G where vertex i is a leaf in T , i.e., $\deg_T(i) = 1$.

Theorem 1.5. *Let $s + t = n$, with $2 \leq s \leq t$, then*

$$\tau(K_{s,t}) = \sum_{i=1}^{t-1} (-1)^{i-1} s^{t-1} (t-i)^{s-1}.$$

Proof. For the graph $K_{s,t}$, let X denote the set of s vertices in the one partite set, Y the vertices in the t -set, i.e., $K_{s,t} = K_{|X|,|Y|}$. Since $2 \leq s \leq t$, i.e., $|X| \leq |Y|$, observe that necessarily any spanning tree T in $K_{s,t}$ must contain a leaf vertex $y \in Y$. This follows since otherwise all vertices $y \in Y$ would then have $\deg_T(y) \geq 2$, and

$$e(T) = \sum_{y \in Y} \deg_T(y) \geq 2|Y| \geq |X| + |Y| = n > n-1,$$

which contradicts that T is a tree with $n-1$ edges. Here $e(T)$ denotes the number of edges in T . Let $Y = \{y_1, y_2, \dots, y_t\}$, then for any spanning tree T in $K_{s,t}$, we have $T \in A_{y_i}$ for some $y_i \in Y$. Consequently $\tau(K_{s,t}) = |A_{y_1} \cup A_{y_2} \cdots \cup A_{y_t}|$. By the principle of the inclusion/exclusion counting formula we have,

$$|A_{y_1} \cup \cdots \cup A_{y_t}| = \sum_{i=1}^{t-1} (-1)^{i-1} \binom{t}{i} \tau(K_{s,t-i}) s^i.$$

Using equation (2) for $\tau(K_{s,t-i})$ gives:

$$\begin{aligned} s^{t-1}t^{s-1} = \tau(K_{s,t}) &= \sum_{i=1}^{t-1} (-1)^i \binom{t}{i} s^{t-i-1} (t-i)^{s-1} s^i \\ &= \sum_{i=1}^{t-1} (-1)^i \binom{t}{i} s^{t-1} (t-i)^{s-1}. \end{aligned}$$

□

Since the LHS and RHS of the equation in Theorem 5 both contain the term s^{t-1} , we obtain the identity:

$$\text{for } 2 \leq s \leq t, \quad t^{s-1} = \sum_{i=1}^{t-1} (-1)^{i-1} (t-i)^{s-1} \binom{t}{i}. \quad (7)$$

With $t = n$ and $p = s - 1$ we rewrite (7) as:

$$n^p = \sum_{i=1}^{n-1} (-1)^{i-1} (n-i)^p \binom{n}{i}, \text{ for integers } 1 \leq p, p < n, n \geq 2. \quad (8)$$

As examples of (8) we have:

$$\begin{aligned} n &= \sum_{i=1}^{n-1} (-1)^{i-1} (n-i) \binom{n}{i}, \quad n \geq 2 \\ n^2 &= \sum_{i=1}^{n-1} (-1)^{i-1} (n-i)^2 \binom{n}{i}, \quad n \geq 3. \end{aligned}$$

For the case of K_n , with $V(K_n) = \{1, \dots, n\}$, and again let A_i be the set of spanning trees T in K_n where vertex i is a leaf in T . We have:

Theorem 1.6. $\tau(K_n) = \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i} (n-i)^{n-2}, n \geq 3.$

Proof. Since any spanning tree T in K_n with $n \geq 3$ must contain a leaf vertex, we have $\tau(K_n) = |A_1 \cup A_2 \cdots \cup A_n|$. Notice that all n vertices cannot be leaves. By the inclusion/exclusion formula we have

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (n-i)^i \tau(K_{n-i}).$$

Replacing $\tau(K_{n-i})$ with equation (1) yields the theorem. □

Applying equation (1) to the LHS of Theorem 1.6 gives the identity:

$$n^{n-2} = \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i} (n-i)^{n-2}. \quad (9)$$

It is interesting that letting $p = n - 2$ in (8) gives (8) = (9). This seems surprising since the motivational arguments come from spanning trees in two different families of graphs, namely, K_n and $K_{s,t}$. However, the connection between the two sets of trees, as indicated in Theorem 1.3, perhaps explains this.

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