

# Binomial Identities Generated by Counting Spanning Trees

T.D. Porter

Department of Mathematics  
Southern Illinois University  
Carbondale, IL 62901-4408  
tporter@math.siu.edu

## Abstract

We partition the set of spanning trees contained in the complete graph  $K_n$  into spanning trees contained in the complete bipartite graph  $K_{s,t}$ . This relation will show that any property of spanning trees in  $K_n$  can be derived from trees in  $K_{s,t}$ . We enumerate the trees in  $K_n$  and  $K_{s,t}$  recursively, and after applying the inclusion/exclusion principle of counting, we obtain some combinatorial and numerical identities. Among them are identities for  $n^p$ , where  $n$  and  $p$  are integers.

*Keywords.* Cayley's Theorem, spanning trees, binomial identities

## 1 Introduction

We use the standard notation and terminology which can be found, e.g., in [9]. Let  $\tau(G)$  denote the number of labelled spanning trees in a graph  $G$ . With  $K_n$  denoting the complete graph of  $n$  vertices and  $K_{s,t}$  the complete bipartite graph with partite sets containing  $s$ , respectively,  $t$  vertices. It is well known [1, 2, 4, 5, 6, 7]:

$$\tau(K_n) = n^{n-2}, \quad n \geq 2 \quad (1)$$

$$\tau(K_{s,t}) = s^{t-1}t^{s-1}, \quad s, t \geq 1. \quad (2)$$

We remark that (1) is often referred to as Cayley's theorem. Let  $s + t = n$ , where  $1 \leq s \leq t$ . We have the following observation:

**Theorem 1.1.** *With  $n \geq 2$ , any spanning tree  $T$  in  $K_n$ , is a spanning tree in  $K_{s,t}$  for a unique pair  $(s, t)$  where  $1 \leq s \leq t$  and  $s + t = n$ .*

*Proof.* Consider any spanning tree  $T$  in  $K_n$ , then  $T$  is a connected bipartite graph and as is well known, necessarily possesses a unique bipartition. So construct this unique bipartition by properly 2-coloring the vertex set of  $T$  with colors red (R) and blue (B). Let the number of red vertices be  $s$  and the number of blue vertices be  $t$ , w.l.o.g. let  $s \leq t$ . We then have  $T \subset K_{s,t}$ .  $\square$

The converse is straightforward.

**Theorem 1.2.** *With  $s+t = n$  any spanning tree in  $K_{s,t}$  is a spanning tree in  $K_n$ .*

*Proof.* This follows since  $K_{s,t} \subset K_n$ .  $\square$

**Theorem 1.3.**  $\tau(K_n) = \sum_{s=1}^{\lfloor n/2 \rfloor} \binom{n}{s} \tau(K_{s,n-s})$ .

*Proof.* By combining Theorems 1.1 and 1.2 we see that to find  $\tau(K_n)$  we can enumerate all labelled spanning trees in the possible  $K_{s,t}$  graphs. With  $1 \leq s \leq t$  and  $s+t = n$ , this is the RHS summation.  $\square$

We rewrite Theorem 1.3 as:

$$\sum_{s=1}^{n-1} \binom{n}{s} \tau(K_{s,n-s}) = 2\tau(K_n). \quad (3)$$

Substituting equations (1) and (2) into (3) yields the identity:

$$\sum_{s=1}^{n-1} \binom{n}{s} s^{n-s-1} (n-s)^{s-1} = 2n^{n-2}. \quad (4)$$

An analytic proof of (4) is forthcoming [3], where we derive the RHS of (4) from the LHS by using Abel's binomial formula. So in this sense, knowledge of  $\tau(K_{s,t})$  implies  $\tau(K_n)$ , yielding a new proof of Cayley's theorem. The ideas in Theorems 1.1, 1.2 are also valid when graphs are unlabelled, since the unique bipartition aspect is a structural property of the graph  $G$ . So, for a connected graph  $G$ , let  $I(G)$ , be the number of non-isomorphic spanning trees in  $G$ , we have:

**Theorem 1.4.**  $I(K_n) = \sum_{s=1}^{\lfloor n/2 \rfloor} I(K_{s,n-s})$ .  $\square$

Observational examples of Theorem 1.4 are:

$$\begin{aligned} I(K_6) &= 6 = I(K_{1,5}) + I(K_{2,4}) + I(K_{3,3}) \\ &= 1 + 2 + 3 \end{aligned}$$

$$\begin{aligned} I(K_7) &= 11 = I(K_{1,6}) + I(K_{2,3}) + I(K_{3,4}) \\ &= 1 + 3 + 7. \end{aligned}$$

Theorem 1.4 suggests a different approach to enumerating  $I(K_n)$  from the usual approach done by Polya and Otter. Since a general formula of  $I(K_{s,t})$  is unknown, it motivates us to research  $I(K_{s,t})$  more deeply.

Getting back to equation (4), one can find a similar formula on Prof. László Székely's home web-page [8], where there is an informative ps, pdf file on Abel's binomial theorem

$$\text{(Székely)} \quad \sum_{s=1}^{n-1} \binom{n}{s} s^{s-1} (n-s)^{n-s-1} = 2(n-1)n^{n-2}. \quad (5)$$

Equating (4) and (5) yields an interesting identity:

$$(n-1) \sum_{s=1}^{n-1} \binom{n}{s} s^{n-s-1} (n-s)^{s-1} = \sum_{s=1}^{n-1} \binom{n}{s} s^{s-1} (n-s)^{n-s-1}. \quad (6)$$

We now derive recursive formulas for  $\tau(K_{s,t})$  and  $\tau(K_n)$  that yield corresponding identities. For a graph  $G$  with vertex set  $V(G) = \{1, 2, \dots, n\}$ , let  $A_i$  denote the set of spanning trees  $T$  in  $G$  where vertex  $i$  is a leaf in  $T$ , i.e.,  $\deg_T(i) = 1$ .

**Theorem 1.5.** *Let  $s + t = n$ , with  $2 \leq s \leq t$ , then*

$$\tau(K_{s,t}) = \sum_{i=1}^{t-1} (-1)^{i-1} s^{t-1} (t-i)^{s-1}.$$

*Proof.* For the graph  $K_{s,t}$ , let  $X$  denote the set of  $s$  vertices in the one partite set,  $Y$  the vertices in the  $t$ -set, i.e.,  $K_{s,t} = K_{|X|,|Y|}$ . Since  $2 \leq s \leq t$ , i.e.,  $|X| \leq |Y|$ , observe that necessarily any spanning tree  $T$  in  $K_{s,t}$  must contain a leaf vertex  $y \in Y$ . This follows since otherwise all vertices  $y \in Y$  would then have  $\deg_T(y) \geq 2$ , and

$$e(T) = \sum_{y \in Y} \deg_T(y) \geq 2|Y| \geq |X| + |Y| = n > n-1,$$

which contradicts that  $T$  is a tree with  $n-1$  edges. Here  $e(T)$  denotes the number of edges in  $T$ . Let  $Y = \{y_1, y_2, \dots, y_t\}$ , then for any spanning tree  $T$  in  $K_{s,t}$ , we have  $T \in A_{y_i}$  for some  $y_i \in Y$ . Consequently  $\tau(K_{s,t}) = |A_{y_1} \cup A_{y_2} \cdots \cup A_{y_t}|$ . By the principle of the inclusion/exclusion counting formula we have,

$$|A_{y_1} \cup \cdots \cup A_{y_t}| = \sum_{i=1}^{t-1} (-1)^{i-1} \binom{t}{i} \tau(K_{s,t-i}) s^i.$$

Using equation (2) for  $\tau(K_{s,t-i})$  gives:

$$\begin{aligned} s^{t-1}t^{s-1} = \tau(K_{s,t}) &= \sum_{i=1}^{t-1} (-1)^i \binom{t}{i} s^{t-i-1} (t-i)^{s-1} s^i \\ &= \sum_{i=1}^{t-1} (-1)^i \binom{t}{i} s^{t-1} (t-i)^{s-1}. \end{aligned}$$

□

Since the LHS and RHS of the equation in Theorem 5 both contain the term  $s^{t-1}$ , we obtain the identity:

$$\text{for } 2 \leq s \leq t, \quad t^{s-1} = \sum_{i=1}^{t-1} (-1)^{i-1} (t-i)^{s-1} \binom{t}{i}. \quad (7)$$

With  $t = n$  and  $p = s - 1$  we rewrite (7) as:

$$n^p = \sum_{i=1}^{n-1} (-1)^{i-1} (n-i)^p \binom{n}{i}, \text{ for integers } 1 \leq p, p < n, n \geq 2. \quad (8)$$

As examples of (8) we have:

$$\begin{aligned} n &= \sum_{i=1}^{n-1} (-1)^{i-1} (n-i) \binom{n}{i}, \quad n \geq 2 \\ n^2 &= \sum_{i=1}^{n-1} (-1)^{i-1} (n-i)^2 \binom{n}{i}, \quad n \geq 3. \end{aligned}$$

For the case of  $K_n$ , with  $V(K_n) = \{1, \dots, n\}$ , and again let  $A_i$  be the set of spanning trees  $T$  in  $K_n$  where vertex  $i$  is a leaf in  $T$ . We have:

**Theorem 1.6.**  $\tau(K_n) = \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i} (n-i)^{n-2}$ ,  $n \geq 3$ .

*Proof.* Since any spanning tree  $T$  in  $K_n$  with  $n \geq 3$  must contain a leaf vertex, we have  $\tau(K_n) = |A_1 \cup A_2 \cdots \cup A_n|$ . Notice that all  $n$  vertices cannot be leaves. By the inclusion/exclusion formula we have

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (n-i)^i \tau(K_{n-i}).$$

Replacing  $\tau(K_{n-i})$  with equation (1) yields the theorem. □

Applying equation (1) to the LHS of Theorem 1.6 gives the identity:

$$n^{n-2} = \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i} (n-i)^{n-2}. \quad (9)$$

It is interesting that letting  $p = n - 2$  in (8) gives (8) = (9). This seems surprising since the motivational arguments come from spanning trees in two different families of graphs, namely,  $K_n$  and  $K_{s,t}$ . However, the connection between the two sets of trees, as indicated in Theorem 1.3, perhaps explains this.

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