Generating Clique-Symmetric Graphs Via Eulerian Digraphs

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Abstract

Let H be a graph of order p, and $\{G_{p1}, G_{p2}, \dots\}$ be a countable sequence of graphs with G_{pn} having pn vertices. The sequence is H-removable if $G_{p1} \cong H$ and $G_{pn} - S \cong G_{p(n-1)}$ where S is any vertex subset of G_{pn} that induces a copy of H. The paper deals with the case $H = K_p$. It provides a construction of graphs with a high degree of symmetry; where for any such graph, all the ways of removing the vertices of any fixed number of disjoint K_p's yields the same subgraph. The authors show such sequences exist by employing eulerian digraphs as generators. The case where each G_{pn} is regular is also studied where directed Cayley-type graphs using a finite group come into play.

Keywords: Cayley, clique, eulerian, isomorphism, reconstruction

1 Introduction

We in general follow the notation in [4]. The digraphs and graphs under consideration are loopless without multiple arcs or edges. We use the notation $[p] = \{1, \dots, p\}$. Call a countable sequence $\{G_{pn}\} = \{G_{p1}, G_{p2}, \dots\}$ of graphs K_p-removable if it satisfies the following two properties:

P1: $G_{p1} \cong K_p$

P2: $G_{pn} - W \cong G_{p(n-1)}$ for every $n \geq 2$ and every vertex subset $W \subset V(G_{pn})$ that induces a K_p in G_{pn}, i.e., $G_{pn}[W] \cong K_p$.

We often right $G_1 = G_2$ in place of $G_1 \cong G_2$.

Let $\vec{D}$ be a digraph of order p, with $d^+(u) = d^-(u)$ for every vertex u in $V(\vec{D})$. Then $\vec{D}$ is an eulerian digraph if $\vec{D}$'s underlying 'undirected' graph is of one component, otherwise $\vec{D}$ is eulerian on each of its underlying components.

Consider a copy of K_p with vertices labelled $\{(1,1), \dots, (p,1)\} = \{(i,1) \mid i \in [p]\}$; call these vertices *vertices at level* 1, and call this graph $D_1(K_p)$. Now consider another copy of K_p with vertices labelled $\{(i,2) \mid i \in [p]\}$, these are vertices at level 2. For any vertex $(i,2)$ join it to vertices $\{(i',1) \mid i' \in N^+(i)\}$ at level 1; we call the graph so formed $D_2(K_p)$.

Now, for any $n \geq 1$, consider the graph which has been constructed level by level, up to n levels, according to the previous definition; call this graph $D_n(K_p)$ or simply D_n when p is clear. We say the digraph $\vec{D}$ *generates* the sequence $\{D_n\}$. In D_n the vertices are of the form (i,j) for every $i \in [p]$ and every $1 \leq j \leq n$, where j is their level; and the edges are of two types:

(i) *fixed-level* edges, say at level j

$$((i_1, j), (i_2, j)) \text{ is an edge for all } i_1, i_2 \in [p] \text{ where } i_1 \neq i_2; \text{ and}$$

(ii) *cross-level* edges, for $j > j'$

$$((i, j), (i', j')) \text{ is an edge if and only if } i' \in N^+(i).$$

For any fixed $i \in [p]$, let $I_i = \{(i,1), \dots, (i,n)\} = \{(i,j) \mid 1 \leq j \leq n\}$ be the set of vertices of D_n in 'column i'. Then, because $i \notin N^+(i)$, i.e., no loops, this is an independent set of vertices. Now let W be a subset of $V(D_n)$ that induces a p-clique, then each of the p independent sets $I_1, \ldots, I_p$ contains exactly one vertex from W.

2 Properties of $\mathrm{K_p}$-removable sequences

We first prove a lemma about the non-existence of K_{p+1}'s in a general K_p-removable sequence $\{G_{pn}\}$. The rest of the section is devoted to the special sequence $\{D_n\}$. We show that $\{D_n\}$ is K_p-removable if $\vec{D}$ is an eulerian digraph. We also give the structure of each non-trivial p-clique in D_n, i.e., p-cliques which contain cross-level edges.

Lemma 2.1. *Suppose the n^{th} member G_{pn} of a K_p-removable sequence contains a K_{p+1}, then G_{pn} is K_{pn}.*

Proof. Without loss of generality, we assume $V(G)$ is partitioned into n p-cliques $L_1, L_2, \ldots, L_n$, so that some vertex v in L_2 is joined to every vertex of L_1. Let x be any vertex in L_1. Deleting the $n-1$ p-cliques $L_3, L_4, \ldots, L_n$, $L_1 + \{v\} - \{x\}$ in this order, we obtain the p-clique $L_2 + \{x\} - \{v\}$. Hence x is adjacent to every vertex of L_2 and the union of L_1 and L_2 is K_{2n}. Consequently, the removal of any $n-2$ disjoint K_p's must produce a K_{2n}. This implies that the union of every two levels L_i and L_j is K_{2n}; therefore, G_{pn} is a complete graph. □

We are interested in the K_p's in D_n. The next theorem gives necessary and sufficient conditions for their existence.

Let $W = \{(1, v_1), \ldots, (p, v_p)\}$ be an arbitrary vertex subset in D_n with exactly one vertex from each independent set I_i. Let W have vertices at m different levels: $\ell_1, \ldots, \ell_m$ where $\ell_1 < \cdots < \ell_m$. For $1 \leq k \leq m$, let $V_k = \{i \,|\, v_i = \ell_k\} \neq \emptyset$ be the set of first coordinates of all vertices of W at level ℓ_k . Then the sets $V_1, \ldots, V_m$ partition $[p] = \{1, \ldots, p\}$.

A W-*skew arc* (x, y), is an arc in $\vec{D}_n$ that joins different level-components of W, i.e., consider two levels, V_i, V_j with $l_i < l_j$ and let $y \in V_i$, $x \in V_j$, then if the edge $e = xy$ is in the induced subgraph $D_n[W]$ of D_n we call the arc (x, y) in our original generating digraph $\vec{D}$ a W-*skew arc*. Let (A, B) denote the set of arcs in D from A to B, i.e., all arcs (a, b) with $a \in A$, $b \in B$.

Theorem 2.2. *With the above notation: a set W of vertices of D_n with level-partition $V_1, V_2, \ldots, V_m$, induces a p-clique iff the associated W-skew arcs form a complete symmetric m-partite subdigraph in $\vec{D}$.*

Proof. Suppose W induces a p-clique. By the definition of D_n, we then have for any $u_j \in V_j$, $u_i \in V_i$ with $l_j > l_i$, that $u_i \in N^+(u_j)$, hence $|(V_j, V_i)| = |V_j||V_i|$. Now, since $d^+(u) = d^-(u)$ for every vertex in $\vec{D}$, it follows that the number of arcs entering any vertex subset equals the number of arcs outgoing from it. Consequently

$$\begin{aligned}\sum_{1\le i<j\le m} |V_i||V_j| &= \sum_{j=2}^{m} |(V_j, V_{j-1} \cup \ldots \cup V_1)| \\ &= \sum_{i=1}^{m-1} |(V_i, V_{i+1} \cup \ldots \cup V_m)| \le \sum_{1\le i<j\le m} |V_i||V_j|.\end{aligned} \tag{1}$$

Now, suppose for some i, j with $i < j$ there are vertices $u_i \in V_i$, $u_j \in V_j$ with $u_j \notin N^+(u_i)$, then

$$\sum_{i=1}^{m-1} |(V_i, V_{i+1} \cup \ldots \cup V_m)| < \sum_{1\le i<j\le m} |V_i||V_j|$$

contradicting (1). Hence, we also have, $u_j \in N^+(u_i)$ for any $u_i \in V_i$, $u_j \in V_j$ and $l_i < l_j$. Consequently, (u_i, u_j) and (u_j, u_i) are arcs in $\vec{D}$.

The converse is straightforward. Each V_i itself induces a $|V_i|$-clique, and with all W-skew arcs present, these cliques are all then joined to form a K_p. □

Corollary 2.3. *Let W be a set of vertices of D_n that induces a p-clique. Then, the number of edges in $D_n - W$ equals the number of edges in D_{n-1}.*

Proof. Consider any vertex (i, j) in D_n. Then $\deg((i, j)) = d^+(i)(n-1)+p-1$. So, if (i, j) is in W then its degree 'outside' W is $d^+(i)(n-1)$, which is independent of its level j. Hence, we may assume W is the K_p introduced at that level n. □

Suppose W induces a p-clique in D_n. Let the vertices of W be $\{(i, w_i) \mid 1 \le i \le p\}$. In the graph $D_n - W$, the set $I_i \backslash \{(i, w_i)\}$ is an independent set; call this the i-th independent set of $D_n - W$. Now we construct a bijection ϕ between the vertices of $D_n - W$ and the vertices of D_{n-1}. Under ϕ, for a fixed $i \in [p]$, the vertices in the i-th independent set of $D_n - W$, namely in the set $I_i \backslash \{(i, w_i)\}$, are to be mapped to the vertices in the i-th independent set of D_{n-1}, namely to the set $\{(i, 1), \ldots, (i, n-1)\}$, as follows:

$$\phi(i,j) = \begin{cases} (i, j-1), & \text{for } w_i < j \leq n \\ (i, j), & \text{for } 1 \leq j < w_i. \end{cases}$$

Clearly ϕ is a bijection. It is straightforward to show ϕ moves edges in $D_n - W$ to edges in D_{n-1}.

Now, from Corollary 2.3, the graphs $D_n - W$ and D_{n-1} have the same number of edges, and so ϕ is an isomorphism. Also, since the sequence $\{D_n\}$ satisfies properties P1 and P2 we have $\{D_n\}$ is K_p-removable, hence:

Theorem 2.4. *Let $\vec{D}$ be any eulerian digraph of order p. Then its generated sequence of graphs $\{D_n\}$ is K_p-removable.* □

Example. 1. $p = 3$, $V(\vec{D}) = \{1, 2, 3\}$, $A(\vec{D}) = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$. Then $\vec{D}$ is eulerian. The first three graphs in the sequence $\{D_n(K_p)\}$ are shown in Fig. 1. Also, notice the bipartition (X, Y) with $X = \{2\}$, $Y = \{1, 3\}$ illustrates Theorem 2.2.

3 Generating *regular* $\mathrm{K_p}$-removable sequences using finite groups

We noted in Cor. 2.3 that the degree of any vertex (i, j) in D_n has degree $\deg((i, j) = d^+(i)(n-1) + p - 1$. In this section, we study the case where the graphs D_n are regular; we use the symbol $\lambda = d^+(i) = d^-(i)$ for every vertex i in $V(\vec{D})$. We construct K_p-removable sequences $\{D_n\}$ for each $0 \leq \lambda \leq p$ and call these (K_p, λ)-removable sequences.

One way to obtain a digraph $\vec{D}$ that generates these sequences is to take a λ-regular 'undirected' graph G on p vertices and 'double' orient each edge xy in G, i.e., replace edge xy with two arcs (x, y) and (y, x), hence creating a directed digraph $\vec{D}$ with $\lambda = d^+(i) = d^-(i)$ for each i. However, this is only sufficient when such a λ-regular G exists. Instead, we use a Cayley-type digraph we obtain from an arbitrary group of order p.

Let $p \geq 3$ and let $\mathcal{G}_p = \{g_1, \ldots, g_p\}$ be a finite group with p elements, where e is the identity element. Let $\Lambda \subseteq \mathcal{G}_p$ be a subset of $\mathcal{G}_p$ with $e \notin \Lambda$ and let $|\Lambda| = \lambda$.

We form a digraph $\vec{D}$ associated with $\mathcal{G}_p$ and Λ as follows:

$$V(\vec{D}) = \{g_1, g_2, \ldots, g_p\} \text{ and}$$
$$(g, g') \text{ is an arc in } \vec{D} \text{ if and only if } g'g^{-1} \in \Lambda.$$

We use the symbol $\vec{D} = (\overrightarrow{\mathcal{G}_p, \Lambda})$ for such an associated digraph. Let $\lambda = |\Lambda|$, then notice $d^+(g_i) = d^-(g_i) = |\Lambda| = \lambda$ for every vertex g_i in $(\overrightarrow{\mathcal{G}_p, \Lambda})$. Hence, the digraph $\vec{D} = (\overrightarrow{\mathcal{G}_p, \Lambda})$ satisfies the eulerian conditions in Section 2, consequently $\vec{D}$ generates a (K_p, λ)-removable sequence $\{D_n\}$, and we have:

Theorem 3.1. *For any $p \geq 3$ and any $\lambda = 0, \ldots, p$, there exists a (K_p, λ)-removable sequence.* □

Example. 2. For the case $\lambda = 0$, resp., $\lambda = p$, we have the unique sequences $\{nK_p\}$, resp., $\{K_{pn}\}$. For $\lambda = 1, \ldots, p-1$, consider the group $\mathcal{G}_p = \{e, g, g^2, \ldots, g^{p-1}\}$ of rotational symmetries of the regular p-gon, where g is the clockwise rotation in the plane through an angle of $360°/p$. With $\lambda = |\Lambda|$, let $\Lambda = \{g, g^2, \ldots, g^{|\Lambda|}\}$, then $(\overrightarrow{\mathcal{G}_p, \Lambda})$ generates (K_p, λ)-removable sequences for each $\lambda = 1, \ldots, p-1$. We remark that for the case $\lambda = p-1$, $(\overrightarrow{\mathcal{G}_p, \Lambda})$ yields the sequence $\{K_{p \times n}\}$, where $K_{p \times n} = K_{\underbrace{n, \ldots, n}_{p}}$ is the complete p-partite graph on pn vertices. It can be shown this is the unique graph for $\lambda = p-1$. Consequently, $(\overrightarrow{\mathcal{G}_p, \Lambda})$ generates a spectrum of graphs associating the three well-known graphs $\{nK_p\}, \ldots, \{K_{p \times n}\}, \{K_{pn}\}$.

We remark that we called $(\overrightarrow{\mathcal{G}_p, \Lambda})$ a Cayley-type digraph. It is a little different than the usual Cayley construction; see, e.g., p. 122 of Biggs [2]. Where there, it is required both:

(i) $e \notin \Lambda$, (ii) $x \in \Lambda \Rightarrow x^{-1} \in \Lambda$. We do not require (ii), since this would create symmetric digraphs $(\overrightarrow{\mathcal{G}_p, \Lambda})$, and we would be back to the same restrictions involving the existence of λ-regular graphs G.

Now consider the (K_p, λ)-removable sequence $\{D_n\}$ obtained from our generating digraph $\vec{D} = (\overrightarrow{\mathcal{G}_p, \Lambda})$. Analogous to Theorem 2.2, we describe the structure of induced p-cliques in D_n.

Let $\overline{\Lambda}$ denote the complement of Λ in $\mathcal{G}_p$ and let $\langle\overline{\Lambda}\rangle$ be the subgroup generated by $\overline{\Lambda}$, also let $\langle\overline{\Lambda}\rangle g$ denote a typical coset of this subgroup.

Let $W = \{(g_1, v_1), \ldots, (g_p, v_p)\}$ be an arbitrary vertex subset in D_n with exactly one vertex from each independent set $I_i = \{(g_i, j) \,|\, 1 \leq j \leq n\}$. As in Section 2, let W have vertices at m different levels: $\ell_1, \ldots, \ell_m$ where $\ell_1 < \cdots < \ell_m$. For $1 \leq k \leq m$, let $V_k = \{g_i \,|\, v_i = \ell_k\} \neq \emptyset$ be the set of first coordinates of all vertices of W at level ℓ_k . Then the sets $V_1, \ldots, V_m$ partition $\mathcal{G}_p$, and we have:

Theorem 3.2. *With the above the notation: a set W of vertices of D_n with level-partition $V_1, \ldots, V_m$ induces a p-clique iff each V_i is a union of cosets of $\langle\overline{\Lambda}\rangle$.*

Proof. Suppose W induces a p-clique. Consider any V_i and let g_i be a vertex in V_i. Then $gg_i \in V_i$ for all $g \in \overline{\Lambda}$, for suppose otherwise; i.e., there exists a $g \in \overline{\Lambda}$ and $gg_i \in V_j$ with $i \neq j$. However, this implies from Theorem 2.2 that (g_i, gg_i) is necessarily an arc in $\vec{D} = (\overrightarrow{\mathcal{G}_p, \Lambda})$, i.e., $(gg_i)g_i^{-1} = g(g_ig_i^{-1}) = g \in \Lambda$, a contradiction. Hence each V_i is a union of cosets relative to $\overline{\Lambda}$.

For the converse, let each V_k be a union of cosets of $\langle\overline{\Lambda}\rangle$. Let (g, ℓ_k) and $(g', \ell_{k'})$ be two arbitrary vertices in W, we show that $((g, \ell_k), (g', \ell_{k'}))$ is an edge in D_n. If $\ell_k = \ell_{k'}$ then, certainly, $((g, \ell_k), (g', \ell_{k'}))$ is an edge by construction of D_n. Otherwise, without loss of generality, let $\ell_k > \ell_{k'}$. Then g and g' are in different cosets of $\langle\overline{\Lambda}\rangle$, so $g'g^{-1} \notin \langle\overline{\Lambda}\rangle$, so $g'g^{-1} \in \overline{\langle\overline{\Lambda}\rangle} \subseteq \Lambda$, and again $((g, \ell_k), (g', \ell_{k'}))$ is an edge. Thus $D_n[W] = K_p$, as required. □

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For other papers on graph sequences see Barefoot, Entringer, and Jackson [1], and the references therein; another somewhat related paper is Duchet, Tuza, and Vestergaard [3].

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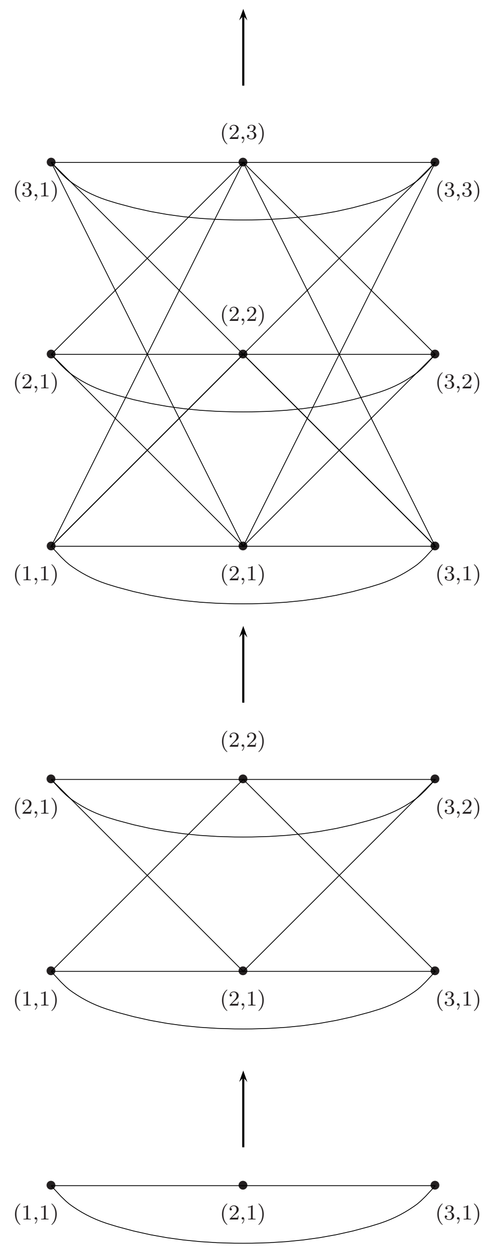

Figure 1