

Clique-Symmetric Uniform Hypergraphs

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Abstract

Let H be an r -uniform hypergraph of order p , and $\{H_{p1}, H_{p2}, \dots\}$ be a countable sequence of r -uniform hypergraphs with H_{pn} having pn vertices. The sequence is H -removable if $H_{p1} \cong H$ and $H_{pn} - S \cong H_{p(n-1)}$ where S is any vertex subset of H_{pn} that induces a copy of H . This paper deals with the case $H = K_p^r$. It provides a construction of hypergraphs with a high degree of symmetry; where for any such hypergraph, all the ways of removing the vertices of any fixed number of disjoint K_p^r 's yields the same subgraph. The case $r = 2$ was studied by the authors in [3]. This paper gives the generalization to r -uniform hypergraphs for all $r = 2, 3, \dots$

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1 The Case $r = 2$

We first briefly list the previous ideas in [3] and [4] concerning the case $r = 2$. The same vocabulary and definitions will be needed in our subsequent sections.

In general we follow the notation in [5]. In particular if W is a subset of the vertices of a graph G , then $G[W]$ denotes the subgraph of G induced by W . The digraphs and graphs we consider are loopless and without multiple arcs or edges. We also use $[p] = \{1, \dots, p\}$.

Call a countable sequence of graphs $\{G_{pn}\} = \{G_{p1}, G_{p2}, \dots\}$ K_p -removable if it satisfies the following two properties:

P1: $G_{p1} \cong K_p$

P2: $G_{pn} - W \cong G_{p(n-1)}$ for every $n \geq 2$ and every vertex subset $W \subset V(G_{pn})$ that induces a K_p in G_{pn} , i.e., $G_{pn} \cong K_p$.

We often write $G_1 = G_2$ in place of $G_1 \cong G_2$.

Let \vec{D} be a digraph of order p , with $d^+(u) = d^-(u)$ for every vertex u in $V(\vec{D})$. Then \vec{D} is an eulerian digraph if \vec{D} 's underlying 'undirected' graph is of one component, otherwise \vec{D} is eulerian on each of its underlying components. Let $N^+(i)$ denote the out-neighborhood of vertex i .

Consider a copy of K_p with vertices labelled $\{(1, 1), \dots, (p, 1)\} = \{(i, 1) \mid i \in [p]\}$; call these vertices *vertices at level 1*, and call this graph $D_1(K_p)$. Now consider another copy of K_p with vertices labelled $\{(i, 2) \mid i \in [p]\}$, these are vertices at level 2. For any vertex $(i, 2)$ join it to the vertices $\{(i', 1) \mid i' \in N^+(i)\}$ at level 1, so we see that these edges are derived from the digraph \vec{D} . We call the graph so formed $D_2(K_p)$. Now consider a third K_p with vertices labelled $\{(i, 3) \mid i \in [p]\}$, at level 3. Join any vertex $(i, 3)$ to vertices $\{(i', 2) \mid i' \in N^+(i)\}$ at level 2 and to vertices $\{(i', 1) \mid i' \in N^+(i)\}$ at level 1; this is $D_3(K_p)$.

Now, for any $n \geq 1$, consider the graph which has been constructed level by level, up to n levels, according to the previous definition; call this graph $D_n(K_p)$ or simply D_n when p is clear. We say the digraph \vec{D} *generates* the sequence $\{D_n\}$. The vertices of D_n , $V(D_n)$, are of the form (i, j) for every $i \in [p]$ and every $1 \leq j \leq n$, where j is their level; and the edges are of two types:

(i) *fixed-level* edges, say at level j

$((i_1, j), (i_2, j))$ is an edge for all $i_1, i_2 \in [p]$ where $i_1 \neq i_2$; and

(ii) *cross-level edges*, for $j > j'$

$((i, j), (i', j'))$ is an edge if and only if $i' \in N^+(i)$.

For any fixed $i \in [p]$, let $I_i = \{(i, 1), \dots, (i, n)\} = \{(i, j) \mid 1 \leq j \leq n\}$ be the set of vertices of D_n in ‘column i ’. Then, because $i \notin N^+(i)$, *i.e.*, because \vec{D} doesn’t have loops, this is an independent set of vertices. Now let W be a subset of $V(D_n)$ that induces a p -clique; then each of the p independent sets I_1, \dots, I_p contain exactly one vertex from W .

Let $W = \{(1, v_1), \dots, (p, v_p)\}$ be an arbitrary vertex subset in D_n with exactly one vertex from each independent set I_i . Let W have vertices at m different levels: ℓ_1, \dots, ℓ_m where $\ell_1 < \dots < \ell_m$. For $1 \leq k \leq m$, let $V_k = \{i \mid v_i = \ell_k\} \neq \emptyset$ be the set of first coordinates of all vertices of W at level ℓ_k . Then the sets V_1, \dots, V_m partition $[p]$.

In D_n consider two levels of vertices, V_i and V_j with $i < j$ and let $y \in V_i$ and $x \in V_j$. Then if the edge $e = xy$ is in the induced subgraph $D_n[W]$ of D_n we call the arc (x, y) in our original generating digraph \vec{D} a *W-skew arc*. Hence a W -skew arc of \vec{D} gives rise to edges in D_n which join different levels of W .

Let (A, B) denote the set of arcs in D from A to B , *i.e.*, all arcs (a, b) with $a \in A$ and $b \in B$.

Theorem 1.1 ([4]). *With the above notation: a set W of vertices of D_n with level-partition V_1, V_2, \dots, V_m , induces a p -clique iff the associated W -skew arcs form a complete symmetric m -partite subdigraph in \vec{D} . \square*

Theorem 1.2 ([4]). *Let \vec{D} be any eulerian digraph of order p . Then its generated sequence of graphs $\{D_n\}$ is K_p -removable. \square*

2 The Hypergraph Construction

The ideas and results in Section 1 lead to the following extension to hypergraphs. A hypergraph consists of a collection of vertices and a collection of edges; if the vertex set is V , then the edges are subsets of V . A hypergraph is r -uniform if all of its edges have size r . The complete r -uniform hypergraph of order p , denoted by K_p^r , is the hypergraph with vertex set $V = [p]$ and with edges all of the $\binom{p}{r}$ r -subsets of V .

For a fixed pair p and r , with $p \geq r \geq 2$, let $\{H_{pn}^r\} = \{H_{p1}^r, H_{p2}^r, H_{p3}^r, \dots\}$ be a sequence of r -uniform hypergraphs where H_{pn}^r has pn vertices. Such a sequence is called K_p^r -removable if it satisfies the following properties:

P1 $H_{p1}^r \cong K_p^r$

P2 $H_{pn}^r - K_p^r \cong H_{p(n-1)}^r$ for every $n \geq 2$ and for every (induced) K_p^r in H_{pn}^r .

For each pair p and r , with $p \geq r \geq 2$, we show the existence of K_p^r -removable sequences.

Let (A_1, \dots, A_m) denote a partition of $[p]$. We define the complete m -partite r -uniform hypergraph of order p , denoted $K_{|A_1|, \dots, |A_m|}^r$, as follows: The vertex set is $[p]$, and an r -subset, Q , of $[p]$ is an edge if and only if Q contains at most $r - 1$ members from any class A_j . So, eg., $K_{n,n}^r$ has $2n$ vertices and

$$\binom{n}{1} \binom{n}{r-1} + \binom{n}{2} \binom{n}{r-2} + \dots + \binom{n}{r-1} \binom{n}{1} = \binom{2n}{r} - 2 \binom{n}{r}$$

edges. Notice that for $r = 2$ we have the usual $K_{n,n}^2$, the complete bipartite graph with n^2 edges. The number of edges in $K_{|A_1|, \dots, |A_m|}^r$ is

$$e(K_{|A_1|, \dots, |A_m|}^r) = \sum \binom{|A_1|}{k_1} \dots \binom{|A_m|}{k_m}$$

where the sum is over all $k_1 + \dots + k_m = r$, with $0 \leq k_j < r$ for all $1 \leq j \leq m$.

Motivated by Theorem 1.1 we now construct K_p^r -removable sequences, $\{H_{pn}^r\}$, from each fixed partition (A_1, \dots, A_m) of $[p]$.

With $\{H_{pn}^r\} = \{H_{p1}^r, H_{p2}^r, \dots\}$ the construction is as follows:

1. $H_{p(m-1)}^r \cong (m-1)K_p^r$, *i.e.*, we first start with $m-1$ disjoint levelled copies of K_p^r .
2. For $n \geq m$, the graph H_{pn}^r is defined as follows. First, take n disjoint levelled copies of K_p^r ; the notation for the vertices introduced at level 1, level 2, \dots , level n , is the same as in Section 1, *eg.*, the copy of K_p^r at level j has vertex set $\{(1, j), (2, j), \dots, (p, j)\}$. From the given partition (A_1, \dots, A_m) of $[p]$, to each A_j , $1 \leq j \leq m$, we select a distinct level l_{A_j} , where $1 \leq l_{A_j} \leq n$. We use $(A_1, \dots, A_m) \rightarrow (l_{A_1}, \dots, l_{A_m})$ to denote the selected levels. Notice that to the partition (A_1, \dots, A_m) there are $m! \binom{n}{m}$ such level $(l_{A_1}, \dots, l_{A_m})$ selections.

To each *fixed* level selection $(A_1, \dots, A_m) \rightarrow (l_{A_1}, \dots, l_{A_m})$ we identify the set A_j with their corresponding vertices in level l_{A_j} . The identification is through the first coordinates of the vertices in l_{A_j} . For example, if $A_j = \{x_1, \dots, x_{|A_j|}\}$, then we identify A_j with the vertices, $\tilde{A}_j = \{(x_1, l_{A_j}), (x_2, l_{A_j}), \dots, (x_{|A_j|}, l_{A_j})\}$ in level l_{A_j} of H_{pn}^r . We then have $(A_1, \dots, A_m) \cong (\tilde{A}_1, \dots, \tilde{A}_m)$ with $|\tilde{A}_1| + \dots + |\tilde{A}_m| = p$, for each of the $m! \binom{n}{m}$ such $(\tilde{A}_1, \dots, \tilde{A}_m)$'s. We then add to the initial n disjoint copies of K_p^r , all edges in $\bigcup E(K_{|\tilde{A}_1|, \dots, |\tilde{A}_m|}^r)$, where the union is over all $m! \binom{n}{m}$ such $(\tilde{A}_1, \dots, \tilde{A}_m)$'s. We call such a graph H_{pn}^r the hypergraph *generated* by the partition (A_1, \dots, A_m) .

We use the symbol $\text{gen}(A_1, \dots, A_m)^r$ to denote such a sequence $\{H_{pn}^r\}$. Notice by the construction, from (A_1, \dots, A_m) and a chosen level- $(\tilde{A}_1, \dots, \tilde{A}_m)$, the vertices $\tilde{A}_1 \cup \dots \cup \tilde{A}_m$ in $V(H_{pn}^r)$ induce a K_p^r . The other K_p^r 's in H_{pn}^r are 'fixed-level' cliques, *i.e.*, the K_p^r introduced at each level $1, \dots, n$. For the vertex subsets I_i , as defined earlier, the construction yields edges which contain at most one vertex from any 'column i '. Hence I_i is an independent set in H_{pn}^r . Also notice that each induced K_p^r contains exactly one vertex from each I_i , $1 \leq i \leq p$.

For $r = 2$, the connection to the digraph \vec{D} in Theorem 1.2 is as follows: Consider any complete m -partite graph $G = K_{|A_1|, \dots, |A_m|}^2$. Let \vec{D} be the digraph formed from G by replacing each edge xy in G with two arcs (x, y) and (y, x) . Then \vec{D} is Eulerian and it generates the K_p^2 -removable sequence $\{D_n\}$ as in Theorem 1.2. Suppose W induces a K_p^r in H_{pn}^r . Let the vertices of W be $\{(i, w_i) \mid 1 \leq i \leq p\}$. In the graph $H_{pn}^r - W$, the set $I_i \setminus \{(i, w_i)\}$ is

an independent set: call this the i -th independent set of $H_{pn}^r - W$. Now we construct an isomorphism ϕ between the vertices of $H_{pn}^r - W$ and the vertices of $H_{p(n-1)}^r$. Under ϕ , for a fixed $i \in [p]$, the vertices in the i -th independent set of $H_{pn}^r - W$, namely in the set $I_i \setminus \{(i, w_i)\}$, are to be mapped to the vertices in the i -th independent set of $H_{p(n-1)}^r$, namely to the set $\{(i, 1), \dots, (i, n-1)\}$, as follows:

$$\phi(i, j) = \begin{cases} (i, j-1), & \text{for } w_i < j \leq n \\ (i, j), & \text{for } 1 \leq j < w_i. \end{cases}$$

In [3] it is shown that ϕ is an isomorphism. It is straightforward to show that ϕ moves edges in $H_{pn}^r - W$ to edges in $H_{p(n-1)}^r$. We have:

Theorem 2.1. *Let $p \geq r \geq 2$, and let (A_1, \dots, A_m) be any partition of $[p]$. Then the sequence of hypergraphs $\text{gen}(A_1, \dots, A_m)^r$ is a K_p^r -removable sequence. \square*

In the following let $S(m, t)$ be the *Stirling numbers of the second kind*.

Notice, by the above construction of $\text{gen}(A_1, \dots, A_m)^r = \{H_{pn}^r\}$, H_{pn}^r contains essentially two types of induced K_p^r 's, either *fixed-level* or *cross-level*. The vertices of a K_p^r are either *all* at a fixed level, or are at *exactly* m different levels. However, by the isomorphism ϕ , the removal of any one of these *two* types of induced K_p^r 's yields up to isomorphism the same sub-hypergraph. So, the hypergraph construction gives the clique-symmetric uniform hypergraph we were searching for that does not simply possess n *fixed-level* type K_p^r 's.

We remark that the $\text{gen}(A_1, \dots, A_m)^r$ construction can be slightly modified to *also* contain *cross-level* K_p^r 's containing vertices from *exactly* t -levels, for all $2 \leq t \leq m-1$, as follows: The partition (A_1, \dots, A_m) of $[p]$ is given. For fixed t , $2 \leq t \leq m-1$, let (b_1, \dots, b_t) be a partition of $[m]$. From (b_1, \dots, b_t) define the corresponding (B_1, \dots, B_t) where $B_j = \bigcup_{k \in b_j} A_k$. Notice that (B_1, \dots, B_t) is then a partition of $[p]$. Also notice that $K_{|B_1|, \dots, |B_t|}^r$ is a complete t -partite sub-hypergraph of $K_{|A_1|, \dots, |A_m|}^r$; and there are $t!S(m, t)$ such possible (b_1, \dots, b_t) 's, and hence $t!S(m, t)$ such possible (B_1, \dots, B_t) 's. We remark that in general, for any complete m -partite graph, where $m \geq 3$, there are $S(m, t)$ complete t -partite spanning subgraphs, where $2 \leq t \leq m-1$. As before, each (B_1, \dots, B_t) is assigned a level-selection $(\ell_{B_1}, \dots, \ell_{B_t})$ producing the corresponding $(\tilde{B}_1, \dots, \tilde{B}_t)$. Now define $X_t = \bigcup E(K_{|\tilde{B}_1|, \dots, |\tilde{B}_t|}^r)$, where the union is over all $(\tilde{B}_1, \dots, \tilde{B}_t)$'s. For the *generalized* construction from a

given (A_1, \dots, A_m) , we again start with n disjoint levelled K_p^r 's. We add to these the edges $X_2 \cup X_3 \cdots \cup X_m$. It can be shown the isomorphism ϕ works for this graph as well.

We call such a K_p^r -removable sequence $\{H_{pn}^r\}$ as constructed above a t -gen $(A_1, \dots, A_m)^r$ sequence. We note that for a given (B_1, \dots, B_t) and its associated $t!(\binom{n}{t})$ corresponding $(\tilde{B}_1, \dots, \tilde{B}_t)$'s, that any permutation $\sigma(B_1, \dots, B_t)$ of (B_1, \dots, B_t) , yields the same set of $(\tilde{B}_1, \dots, \tilde{B}_t)$'s as (B_1, \dots, B_t) does. Hence there are $t!S(m, t)\binom{n}{t}$ such $(\tilde{B}_1, \dots, \tilde{B}_t)$'s, for each $2 \leq t \leq m$. By the construction every K_p^r in H_{pn}^r is in one-to-one correspondence with a fixed $(\tilde{B}_1, \dots, \tilde{B}_t)$. For the case $t = 1$, we interpret the (\tilde{B}_1) 's as the beginning n -level copies of K_p^r , so $X_1 \cong nK_p^r$. Hence we have a rather nice formula for the number of K_p^r 's in H_{pn}^r .

Theorem 2.2. *Let (A_1, \dots, A_m) be a partition of $[p]$. For the K_p^r -removable sequence t -gen $(A_1, \dots, A_m)^r = \{H_{pn}^r\}$, the hypergraph H_{pn}^r has $\sum_{t=1}^m t!S(m, t)\binom{n}{t} = n^m$ K_p^r 's. \square*

More directly, the value n^m in Theorem 2.2 can be interpreted as follows: From the given initial partition (A_1, \dots, A_m) of $[p]$ we embed each A_j (by our definition of \tilde{A}_j) into any of the n levels, level 1, \dots , level n , thus creating $(\tilde{B}_1, \dots, \tilde{B}_t)$. Hence there are n^m such embeddings. We remark also that, by the defined construction, for each embedding $(\tilde{B}_1, \dots, \tilde{B}_t)$ the vertices $\tilde{B}_1 \cup \dots \cup \tilde{B}_t$ induce a K_p^r in H_{pn}^r .

Example Let $p = 5$, $r = 3$, and (A_1, A_2, A_3) be the partition of $\{1, 2, 3, 4, 5\}$ with $A_1 = \{1, 2\}$, $A_2 = \{3, 5\}$, and $A_3 = \{4\}$. Consider the t -gen $(A_1, A_2, A_3)^3 = \{H_{5n}^3\}$ sequence. The graph for $n = 3$, *i.e.*, $H_{(5)(3)}^3$ has $1!S(3, 1)\binom{3}{1} + 2!S(3, 2)\binom{3}{2} + 3!S(3, 3)\binom{3}{3} = 27$ induced K_5^3 's. An example of one of these cliques for $t = 2$ is as follows: with $(B_1, B_2) = (A_1 \cup A_3, A_2)$, let $(B_1, B_2) \rightarrow (\ell_{B_1}, \ell_{B_2}) = (3, 2)$ be a level-selection, giving $(\tilde{B}_1, \tilde{B}_2)$, then the vertices in $V(H_{(5)(3)}^3)$, namely, $\tilde{B}_1 \cup \tilde{B}_2 = \{(3, 1), (3, 2), (3, 4), (2, 3), (2, 5)\}$, induce a K_5^3 in $H_{(5)(3)}^3$.

K_{pn}^r is the complete r -uniform hypergraph on pn vertices.

We end by showing, with the exception of $\{K_{pn}^r\}$, that for any K_p^r -removable sequence $\{H_{pn}^r\}$, with $p \geq r \geq 2$, any member H_{pn}^r of the sequence does not contain a K_{p+1}^r . So, *e.g.*, for the case $r = 2$, the clique number is always $\omega(H_{pn}^2) = p$ for all n . We remark that the constructions given in this paper produce ways to generate K_p^r -removable sequences, we do not

claim these are the *only* K_p^r -removable sequences. However, our final theorem applies to *any* K_p^r -removable sequence.

Theorem 2.3. *Suppose the n -th member H_{pn}^r of a K_p^r -removable sequence contains a K_{p+1}^r , then H_{pn}^r is K_{pn}^r .*

Proof. Suppose that H_{pn}^r contains a K_{p+1}^r . Without loss of generality, we assume $V(H_{pn}^r)$ is partitioned into n K_p^r 's: L_1, L_2, \dots, L_n , so that some vertex y in L_2 is such that $L_1 \cup \{y\}$ is an induced K_{p+1}^r in H_{pn}^r . Let x be any vertex in L_1 . Deleting the $n - 1$ K_p^r 's: $L_3, L_4, \dots, L_n, L_1 + \{y\} - \{x\}$, in this order, we obtain $L_2 + \{x\} - \{y\}$. Since $\{H_{pn}^r\}$ is K_p^r -removable, $L_2 + \{x\} - \{y\}$ is necessarily a K_p^r . Hence, $L_2 \cup \{x\}$ is a K_{p+1}^r , and the union of L_1 and L_2 is K_{2p}^r . Consequently the removal of any $n - 2$ disjoint K_p^r 's must produce K_{2p}^r . This implies that the union of every two levels L_i and L_j induce a K_{2p}^r ; therefore, H_{pn}^r is K_{pn}^r . \square

For other papers on graph sequences see Barefoot, Entringer, and Jackson [1], and its bibliography; see also Duchet, Tuza, and Vestergaard [2].

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