

# $K_p$ -Removable Sequences of Graphs

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### Abstract

Let  $\{G_{pn} \mid n \geq 1\} = \{G_{p1}, G_{p2}, G_{p3}, \dots\}$  be a countable sequence of simple graphs, where  $G_{pn}$  has  $pn$  vertices. This sequence is called  $K_p$ -removable if  $G_{p1} = K_p$ , and  $G_{pn} - K_p = G_{p(n-1)}$  for every  $n \geq 2$  and for every  $K_p$  in  $G_{pn}$ . We give a general construction of such sequences. We specialize to sequences in which each  $G_{pn}$  is regular; these are called regular  $(K_p, \lambda)$ -removable sequences,  $\lambda$  is a fixed number,  $0 \leq \lambda \leq p$ , referring to the fact that  $G_{pn}$  is  $(\lambda(n-1) + p - 1)$ -regular. We classify regular  $(K_p, 0)$ -,  $(K_p, p - 1)$ -, and  $(K_p, p)$ -removable sequences as the sequences  $\{nK_p \mid n \geq 1\}$ ,  $\{K_{p \times n} \mid n \geq 1\}$ , and  $\{K_{pn} \mid n \geq 1\}$  respectively. Regular sequences are also constructed using ‘levelled’ Cayley graphs, based on a finite group. Some examples are given.

*Keywords:* removable, Cayley, isomorphism, reconstruction, clique

# 1 Notation, $K_p$ -removable sequences of graphs, main results

For  $p \geq 1$  and  $n \geq 1$  let  $K_p$  be the complete graph on  $p$  vertices, let  $nK_p$  be  $n$  disjoint copies of  $K_p$ , and let  $K_{p \times n} = \underbrace{K_{n, \dots, n}}_p$ , be the

complete  $p$ -partite graph on  $pn$  vertices. All graphs  $G$  are simple. In the graph  $G$  let  $G[\{v_1, \dots, v_m\}] = G[v_1, \dots, v_m]$  denote the induced subgraph on vertices  $\{v_1, \dots, v_m\}$ . Suppose that for some  $p$  vertices  $\{v_1, \dots, v_p\}$  in  $G$  we have  $G[v_1, \dots, v_p] = K_p$ , i.e.,  $G[v_1, \dots, v_p]$  is an induced  $K_p$ , then  $G - K_p$  denotes the subgraph obtained from  $G$  by deleting vertices  $v_i$  and their incident edges, for every  $i = 1, \dots, p$ . If two graphs  $G$  and  $G'$  are isomorphic we write  $G \cong G'$ . We often say that two graphs are ‘equal’ ( $=$ ) instead of ‘isomorphic’, and say ‘ $K_p$ ’ instead of ‘induced  $K_p$ ’.

For the countable sequence of graphs  $\{G_{pn} \mid n \geq 1\} = \{G_{p1}, G_{p2}, G_{p3}, \dots\}$  we use the notation  $\{G_{pn}\}$ , each graph  $G_{pn}$  has  $pn$  vertices for a fixed  $p \geq 1$ .

We call a sequence  $\{G_{pn}\}$   $K_p$ -removable if it satisfies the following two properties:

**A1**  $G_{p1} \cong K_p$ ,

**A2**  $G_{pn} - K_p \cong G_{p(n-1)}$  for every  $n \geq 2$  and for every (induced)  $K_p$  in  $G_{pn}$ .

In this paper we investigate  $K_p$ -removable sequences. In Section 2 we give a general construction for such sequences. In Section 3 we specialize to sequences in which each graph is regular; we call these regular  $(K_p, \lambda)$ -removable sequences,  $\lambda$  is a fixed number,  $0 \leq \lambda \leq p$ , referring to the fact that  $G_{pn}$  is  $(\lambda(n-1) + p - 1)$ -regular. We classify regular  $(K_p, 0)$ -,  $(K_p, p-1)$ -, and  $(K_p, p)$ -removable sequences as the sequences  $\{nK_p\}$ ,  $\{K_{p \times n}\}$ , and  $\{K_{pn}\}$  respectively, thus associating three

well-known graphs on  $pn$  vertices. In Section 4 we construct regular  $(K_p, \lambda)$ -removable sequences starting from a finite group; the graphs in this sequence are similar in construction to Cayley graphs, we also count the number of  $K_p$ 's in these graphs.

## 2 Construction of a $K_p$ -removable sequence for every $p \geq 3$ , examples

From **A1** and **A2** above we see that if  $\{G_{pn}\}$  is  $K_p$ -removable then  $G_{p1} = K_p$  and  $G_{p2}$  is the union of this  $K_p$  and a 'new'  $K_p$ , together with some edges between them. The graph  $G_{p3}$  is then formed from  $G_{p2}$  by adding another  $K_p$  and some suitable set of edges between this new  $K_p$  and the previous two  $K_p$ 's, and so on. Thus  $G_{pn}$  contains at least  $n$  disjoint  $K_p$ 's. We use this idea of constructing  $G_{pn}$  by adding  $K_p$  to  $K_p$ , up to  $n$   $K_p$ 's, in the following:

Here  $p \geq 3$  and  $[p] = \{1, \dots, p\}$ . Let  $\Lambda = (\Lambda_1, \dots, \Lambda_p)$  be an ordered  $p$ -tuple of subsets of  $[p]$ , *i.e.*, each  $\Lambda_i \subseteq [p]$ .

Consider a copy of  $K_p$  with vertices labelled  $\{(1, 1), \dots, (p, 1)\} = \{(i, 1) \mid i \in [p]\}$ ; call these vertices *vertices at level 1*, and call this graph  $H_{p1}^\Lambda$ . Now consider another  $K_p$  with vertices labelled  $\{(i, 2) \mid i \in [p]\}$ , vertices at level 2. For any vertex  $(i, 2)$  join it to vertices  $\{(i', 1) \mid i' \in \Lambda_i\}$  at level 1; call this graph  $H_{p2}^\Lambda$ . Now consider a third  $K_p$  with vertices labelled  $\{(i, 3) \mid i \in [p]\}$ , at level 3. Join any vertex  $(i, 3)$  to vertices  $\{(i', 2) \mid i' \in \Lambda_i\}$  at level 2 and to vertices  $\{(i', 1) \mid i' \in \Lambda_i\}$  at level 1; this is  $H_{p3}^\Lambda$ .

For any  $n \geq 1$ , consider the graph which has been constructed level by level, up to  $n$  levels, according to the above definition; call this graph  $H_{pn}^\Lambda$ . In  $H_{pn}^\Lambda$  the vertices are of the form  $(i, j)$  for every  $i \in [p]$  and every  $1 \leq j \leq n$ , (where  $j$  is their level); and the edges are of two types:

(i) *fixed-level* edges, say at level  $j$

$((i_1, j), (i_2, j))$  is an edge for all  $i_1, i_2 \in [p]$  where  $i_1 \neq i_2$ ; and

(ii) *cross-level* edges, for  $j > j'$

$((i, j), (i', j'))$  is an edge if and only if  $i' \in \Lambda_i$ .

For each  $i \in [p]$  let  $\lambda_i = |\Lambda_i|$  be the number of elements in  $\Lambda_i$ , and let  $\mu_i$  be the number of sets in  $\Lambda$  which contain  $i$ . Call  $\Lambda$  *uniform* if:

$$i \notin \Lambda_i \text{ and } \lambda_i = \mu_i \text{ for each } i \in [p]. \quad (1)$$

From now on let our  $\Lambda$  be uniform. In Theorem 2.3 we show that if  $\Lambda$  is uniform then  $\{H_{pn}^\Lambda\}$  is  $K_p$ -removable.

For any fixed  $i \in [p]$ , let  $I_i = \{(i, 1), \dots, (i, n)\} = \{(i, j) \mid 1 \leq j \leq n\}$  be the set of vertices of  $H_{pn}^\Lambda$  in ‘column  $i$ ’. Then, because  $i \notin \Lambda_i$ , this is an independent set of vertices.

Now let  $W$  be a  $K_p$  in  $H_{pn}^\Lambda$ , then each of the  $p$  independent sets  $I_1, \dots, I_p$  contains exactly one vertex from  $W$ ; let  $I_i$  contain vertex  $(i, w_i) \in W$ , a vertex at level  $w_i$ , for some  $1 \leq w_i \leq n$ . Thus  $W = H_{pn}^\Lambda[(1, w_1), \dots, (p, w_p)] = K_p$ .

**Lemma 2.1** *Let  $\Lambda$  be uniform. For any  $K_p = W$  in  $H_{pn}^\Lambda$  the number of edges in  $H_{pn}^\Lambda - W$  equals the number of edges in  $H_{p(n-1)}^\Lambda$ .*

*Proof.* Consider any vertex  $(i, j)$  in  $H_{pn}^\Lambda$ , here  $1 \leq j \leq n$ . It is adjacent to  $\lambda_i$  vertices at each of the  $j - 1$  levels lower than level  $j$ , *i.e.*, to  $\lambda_i(j - 1)$  such vertices, and to  $p - 1$  vertices at level  $j$ , and to  $\mu_i(n - j)$  vertices at levels higher than level  $j$ . Thus, because  $\lambda_i = \mu_i$ , its degree is

$$\deg((i, j)) = \lambda_i(n - 1) + p - 1. \quad (2)$$

So if  $(i, j)$  is in  $W = K_p$ , then its degree ‘outside’  $W$  is  $\lambda_i(n - 1)$ , which is independent of its level,  $j$ .

Now  $W$  contains exactly one vertex from each independent set  $I_i$ , so, when removing  $W$ , we remove  $\sum_{i=1}^p (\lambda_i(n-1))$  edges ‘outside’  $W$ , (and  $\binom{p}{2}$  ‘inside’  $W$ ). But this equals the number of edges removed if we remove the  $K_p$  at level  $n$  (because  $\deg((i, n))$  is also given by (2)), leaving the graph  $H_{p(n-1)}^\Lambda$ . Hence, the number of edges in  $H_{pn}^\Lambda - W$  equals the number of edges in  $H_{p(n-1)}^\Lambda$ . ■

**Lemma 2.2** *Let  $\Lambda$  be uniform. For any  $p \geq 3$ ,  $n \geq 2$ , and any  $K_p = W$  in  $H_{pn}^\Lambda$ , we have*

$$H_{pn}^\Lambda - W = H_{p(n-1)}^\Lambda.$$

*Proof.* Let the vertices of  $W$  be  $\{(i, w_i) \mid 1 \leq i \leq p\}$ . We construct a bijection  $\phi$  between the vertices of  $H_{pn}^\Lambda - W$  and the vertices of  $H_{p(n-1)}^\Lambda$ , and then show that  $\phi$  is an isomorphism. Under  $\phi$ , for a fixed  $i \in [p]$ , the vertices in the  $i$ -th independent set of  $H_{pn}^\Lambda - W$ , namely in the set  $I_i \setminus \{(i, w_i)\}$ , are mapped to the vertices in the  $i$ -th independent set of  $H_{p(n-1)}^\Lambda$ , namely to the set  $\{(i, 1), \dots, (i, n-1)\}$ , as follows:

$$\begin{array}{ccc} (i, n) & \rightarrow & (i, n-1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ (i, w_i + 1) & \rightarrow & (i, w_i) \\ (i, w_i - 1) & \rightarrow & (i, w_i - 1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ (i, 1) & \rightarrow & (i, 1) \end{array}$$

*i.e.,*

$$\phi(i, j) = \begin{cases} (i, j-1), & \text{for } w_i < j \leq n; \\ (i, j), & \text{for } 1 \leq j < w_i. \end{cases}$$

Clearly  $\phi$  is a bijection.

There are two types of edges in  $H_{pn}^\Lambda$ : fixed-level edges and cross-level edges. First we deal with fixed-level edges.

A typical fixed-level edge in  $H_{pn}^\Lambda$  is  $((i_1, j), (i_2, j))$  for some  $i_1, i_2 \in [p]$  where  $i_1 \neq i_2$  and for some  $j$  with  $1 \leq j \leq n$ . Thus, a typical fixed-level edge in  $H_{pn}^\Lambda - W$  is  $((i_1, j), (i_2, j))$  where  $i_1 \neq i_2$ , and  $j \neq w_{i_1}$  and  $j \neq w_{i_2}$ , (because the vertices  $(i_1, w_{i_1})$  and  $(i_2, w_{i_2})$  have been removed). Without loss of generality let  $w_{i_1} \leq w_{i_2}$ .

Now we check that  $\phi$  maps two adjacent vertices at level  $j$  in  $H_{pn}^\Lambda - W$  onto two adjacent vertices in  $H_{p(n-1)}^\Lambda$ . There are three cases to consider:

- (a)  $1 \leq j < w_{i_1} \leq w_{i_2} \leq n$ . Then  $\phi((i_1, j), (i_2, j)) = (\phi(i_1, j), \phi(i_2, j)) = ((i_1, j), (i_2, j))$ , which is certainly a (fixed-level) edge in  $H_{p(n-1)}^\Lambda$ .
- (b)  $1 \leq w_{i_1} < j < w_{i_2} \leq n$ . Then  $\phi((i_1, j), (i_2, j)) = ((i_1, j-1), (i_2, j))$ . Now  $w_{i_1} < w_{i_2}$  and  $((i_1, w_{i_1}), (i_2, w_{i_2}))$  is an edge in  $W$ , and so in  $H_{pn}^\Lambda$ , so  $i_1 \in \Lambda_{i_2}$ ; and  $1 \leq j-1 < j \leq n-1$  and so  $((i_1, j-1), (i_2, j))$  is a (cross-level) edge in  $H_{p(n-1)}^\Lambda$ .
- (c)  $1 \leq w_{i_1} \leq w_{i_2} < j \leq n$ . Then  $\phi((i_1, j), (i_2, j)) = ((i_1, j-1), (i_2, j-1))$ , which, again, is a fixed-level edge in  $H_{p(n-1)}^\Lambda$ .

Cross-level edges in  $H_{pn}^\Lambda$  are of the form  $((i, j), (i', j'))$  where  $j > j'$  and  $i' \in \Lambda_i$ . Thus cross-level edges in  $H_{pn}^\Lambda - W$  are of the form  $((i, j), (i', j'))$ , where  $j > j'$ ,  $j \neq w_i$ ,  $j' \neq w_{i'}$ , and  $i' \in \Lambda_i$ .

Now we check that  $\phi$  maps two adjacent vertices at levels  $j$  and  $j'$  in  $H_{pn}^\Lambda - W$  onto two adjacent vertices in  $H_{p(n-1)}^\Lambda$ . There are four cases to consider:

- (a)  $1 \leq j < w_i \leq n$  and  $1 \leq j' < w_{i'} \leq n$ . Then  $\phi((i, j), (i', j')) = ((i, j), (i', j'))$ , a cross-level edge in  $H_{p(n-1)}^\Lambda$ .
- (b)  $1 \leq j < w_i \leq n$  and  $1 \leq w_{i'} < j' \leq n$ . Then  $\phi((i, j), (i', j')) = ((i, j), (i', j'-1))$ , again, a cross-level edge in  $H_{p(n-1)}^\Lambda$ .
- (c)  $1 \leq w_i < j \leq n$  and  $1 \leq j' < w_{i'} \leq n$ . Then  $\phi((i, j), (i', j')) = ((i, j-1), (i', j'))$ ; here  $j-1 \geq j'$ . If  $j-1 > j'$  then this is a cross-level edge in  $H_{p(n-1)}^\Lambda$ , or, if  $j-1 = j'$ , then this is a fixed-level edge in  $H_{p(n-1)}^\Lambda$ .

(d)  $1 \leq w_i < j \leq n$  and  $1 \leq w_{i'} < j' \leq n$ . Then  $\phi((i, j), (i', j')) = ((i, j-1), (i', j'-1))$ , a cross-level edge in  $H_{p(n-1)}^\Lambda$ .

Thus  $\phi$  moves edges in  $H_{pn}^\Lambda - W$  to edges in  $H_{p(n-1)}^\Lambda$ .

Now, from Lemma 2.1, the graphs  $H_{pn}^\Lambda - W$  and  $H_{p(n-1)}^\Lambda$  have the same number of edges, and so  $\phi$  is an isomorphism. ■

Thus we have the following existence result for  $K_p$ -removable sequences:

**Theorem 2.3** *Let  $\Lambda$  be uniform. For any  $p \geq 3$  the sequence  $\{H_{pn}^\Lambda\}$  is  $K_p$ -removable.*

*Proof.* By construction, for every  $n \geq 1$ , the graph  $H_{pn}^\Lambda$  has  $pn$  vertices. Clearly the sequence  $\{H_{pn}^\Lambda\}$  satisfies **A1**, and, from Lemma 2.2, it satisfies **A2**, hence it is  $K_p$ -removable. ■

**Example 1**  $p = 3$ ,  $\Lambda_1 = \{2\}$ ,  $\Lambda_2 = \{1, 3\}$ ,  $\Lambda_3 = \{2\}$ . Here  $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$  is uniform with  $\lambda_1 = \mu_1 = 1$ ,  $\lambda_2 = \mu_2 = 2$ , and  $\lambda_3 = \mu_3 = 1$ . The first 3 graphs in the  $K_3$ -removable sequence  $\{H_{3n}^\Lambda\}$  are shown in Fig. 1.

The converse of Theorem 2.3 is not true, consider the example:

**Example 2**  $p = 3$ ,  $\Lambda_1 = \{2\}$ ,  $\Lambda_2 = \Lambda_3 = \emptyset$ . Here  $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ , and it is straightforward to show that  $\{H_{3n}^\Lambda\}$  is  $K_3$ -removable but  $\lambda_1 = 1$  and  $\mu_1 = 0$ , and so  $\lambda_1 \neq \mu_1$  and  $\Lambda$  is not uniform.

We call a  $K_p$ -removable sequence  $\{G_{pn}\}$  *regular* if all graphs in the sequence are regular, and *irregular* if at least one graph in the sequence is irregular.

In Section 3 we show that all  $K_p$ -removable sequences for  $p = 1$  and  $p = 2$  are regular. As the next example shows, an irregular  $K_p$ -removable sequence exists for every  $p \geq 3$ .

**Example 3** For a fixed  $p \geq 3$ ,  $\Lambda_1 = \{2\}$ ,  $\Lambda_2 = \{1, 3\}$ ,  $\Lambda_3 = \{2\}$ , and  $\Lambda_i = \emptyset$  for  $4 \leq i \leq p$ . Then  $\Lambda = (\Lambda_1, \dots, \Lambda_p)$  is uniform

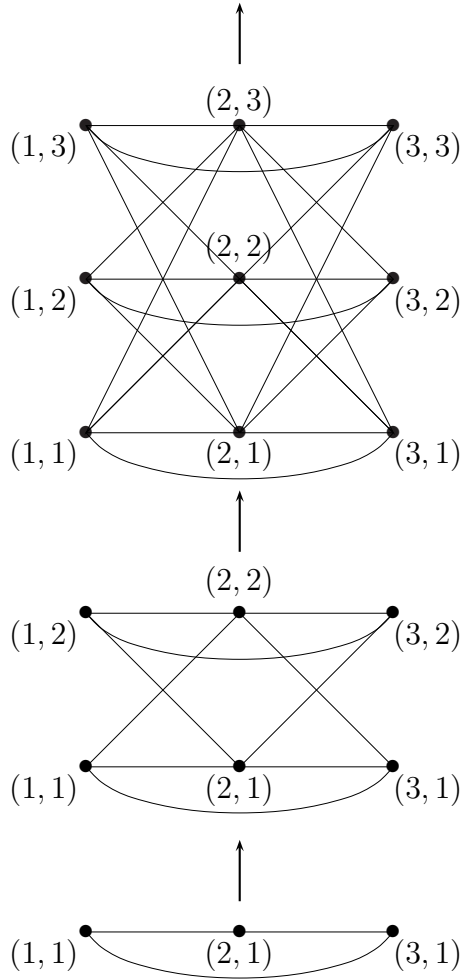


Figure 1: The first 3 graphs in the irregular  $K_3$ -removable sequence  $\{H_{3n}^\Lambda\}$  where  $\Lambda = \{\Lambda_1, \Lambda_2, \Lambda_3\}$  with  $\Lambda_1 = \{2\}$ ,  $\Lambda_2 = \{1, 3\}$ , and  $\Lambda_3 = \{2\}$ . See Examples 1 and 3.

and so  $\{H_{pn}^\Lambda\}$  is  $K_p$ -removable. Moreover,  $\{H_{pn}^\Lambda\}$  is irregular because  $H_{p2}^\Lambda$  is irregular:  $\deg((1, 2)) = p$  but  $\deg((2, 2)) = p + 1$ . See Fig. 1 where  $p = 3$ ,  $\deg((1, 2)) = 3$  but  $\deg((2, 2)) = 4$ .

We are interested in the  $K_p$ 's in  $H_{pn}^\Lambda$ . The next theorem gives necessary and sufficient conditions for their existence.

Let  $V = H_{pn}^\Lambda[(1, v_1), \dots, (p, v_p)]$  be an arbitrary induced subgraph in  $H_{pn}^\Lambda$  with exactly one vertex from each independent set  $I_i$ . Let  $V$  have vertices at  $m$  different levels:  $\ell_1, \dots, \ell_m$  where  $\ell_1 < \dots < \ell_m$ . For  $1 \leq k \leq m$ , let  $V_k = \{i \mid v_i = \ell_k\} \neq \emptyset$  be the set of first coordinates of all vertices of  $V$  at level  $\ell_k$ . Then the sets  $V_1, \dots, V_m$  partition  $[p] = \{1, \dots, p\}$ , and:

**Theorem 2.4** *With the above notation  $V = K_p$  if and only if for every  $k$ , with  $1 \leq k \leq m$ , we have*

$$V_1 \cup \dots \cup V_k = \bigcap_{i \in V_{k+1} \cup \dots \cup V_m} \Lambda_i,$$

where we define  $\bigcap_{i \in \emptyset} \Lambda_i = [p]$ .

*Proof.* Suppose that  $V = K_p$ , and suppose  $i' \in V_1 \cup \dots \cup V_k$  for some  $k$  with  $1 \leq k \leq m$ . Now vertex  $(i', v_{i'})$  is at a lower level than all vertices  $(i, v_i)$  where  $i \in V_{k+1} \cup \dots \cup V_m$ , so  $i' \in \Lambda_i$  for all  $i \in V_{k+1} \cup \dots \cup V_m$ . Thus  $i' \in \bigcap_{i \in V_{k+1} \cup \dots \cup V_m} \Lambda_i$ , and  $V_1 \cup \dots \cup V_k \subseteq \bigcap_{i \in V_{k+1} \cup \dots \cup V_m} \Lambda_i$ . But, because,  $i \notin \Lambda_i$ , we have  $\bigcap_{i \in V_{k+1} \cup \dots \cup V_m} \Lambda_i \subseteq [p] \setminus \{V_{k+1} \cup \dots \cup V_m\} = V_1 \cup \dots \cup V_k$ . Hence  $V_1 \cup \dots \cup V_k = \bigcap_{i \in V_{k+1} \cup \dots \cup V_m} \Lambda_i$ , as required. The converse is straightforward. ■

**Example 4** (See Example 1 and Fig. 1). Here  $p = 3$ ,  $\Lambda_1 = \{2\}$ ,  $\Lambda_2 = \{1, 3\}$ , and  $\Lambda_3 = \{2\}$ . In Fig. 1 consider the induced graph  $H_{33}^\Lambda[(1, 1), (2, 3), (3, 1)] = K_3$ . This  $K_3$  has vertices at  $m = 2$  levels, *i.e.*, at level  $\ell_1 = 1$  and at level  $\ell_2 = 3$ , so  $V_1 = \{1, 3\}$  and  $V_2 = \{2\}$ . For  $k = 1$  we have

$$V_1 = \bigcap_{i \in V_2} \Lambda_i = \Lambda_2,$$

and for  $k = 2$

$$V_1 \cup V_2 = \bigcap_{i \in \emptyset} \Lambda_i = [3] = \{1, 2, 3\}.$$

Such a  $K_p$  with vertices at more than one level is a *cross-level*  $K_p$ .

### 3 Regular sequences, uniqueness of $(K_p, \lambda)$ -removable sequences for $\lambda = 0, p - 1, \text{ or } p$

Here we consider regular  $K_p$ -removable sequences  $\{G_{pn}\}$ , *i.e.*, those which also satisfy:

**A3**  $G_{pn}$  is regular for every  $n \geq 1$ .

For such a sequence we use the notation  $\{R_{pn}\}$ , for every  $n \geq 1$   $R_{pn}$  is regular of degree  $\deg(R_{pn})$ . We know that  $R_{p1} = K_p$ , and the second graph in this sequence is  $R_{p2}$ , so let us define  $\lambda$  by

$$\lambda = \deg(R_{p2}) - \deg(R_{p1}) = \deg(R_{p2}) - (p - 1). \quad (3)$$

**Lemma 3.1** *For a fixed  $p \geq 1$  let  $\{R_{pn}\}$  be  $K_p$ -removable with  $\lambda$  defined as above. Then*

- (i)  $0 \leq \lambda \leq p$ ,
- (ii)  $\deg(R_{pn}) = \lambda(n - 1) + p - 1$  for every  $n \geq 1$ ,
- (iii)  $\lambda = \deg(R_{pn}) - \deg(R_{p(n-1)})$  for every  $n \geq 2$ .

*Proof.* (i) By **A1** and **A2** the graph  $R_{p2}$  contains a  $K_p$  with  $R_{p2} - K_p = R_{p1} = K_p$ . By **A3**  $R_{p2}$  is regular and  $\deg(R_{p2}) \geq \deg(K_p) = p - 1$ , and so  $\lambda = \deg(R_{p2}) - (p - 1) \geq 0$ . Also,  $\deg(R_{p2}) \leq \deg(R_{p1}) + p$  because the degree of a vertex in  $R_{p1}$  can be increased by at most  $p$  when constructing  $R_{p2}$  (if it is made adjacent to each of the  $p$  vertices in the new  $K_p$ ), hence  $\lambda \leq p$ . Thus  $0 \leq \lambda \leq p$ , *i.e.*,  $\lambda = 0, \dots, p$ .

(ii) Here we use induction on  $n$ . For  $n = 1$  (ii) is true by **A1**, and for  $n = 2$  it is true by (3). So assume that  $n \geq 3$  and that (ii) is true for  $n - 1$ , *i.e.*, for the graph  $R_{p(n-1)}$ . Now  $R_{pn}$  is obtained from  $R_{p(n-1)}$  by adding on a new  $K_p$  and some cross-level edges between them. But  $R_{pn}$  is regular, so we can count these cross-level edges in two different ways:

$$(n - 1)p [\deg(R_{pn}) - \deg(R_{p(n-1)})] = p [\deg(R_{pn}) - (p - 1)].$$

This gives

$$\begin{aligned} \deg(R_{pn}) &= \frac{(n - 1)\deg(R_{p(n-1)}) - (p - 1)}{n - 2} \\ &= \lambda(n - 1) + p - 1, \end{aligned}$$

using the induction hypothesis. Thus the induction goes through and (ii) is true for every  $n \geq 1$ .

(iii) This follows directly from (ii). ■

For a fixed  $p \geq 1$  and a fixed  $\lambda$  where  $\lambda = 0, \dots, p$ , let  $\{R_{pn}\}$  be a  $K_p$ -removable sequence in which  $\deg(R_{pn}) = \lambda(n - 1) + p - 1$  for every  $n \geq 1$ . We call the sequence  $\{R_{pn}\}$  *regular*  $(K_p, \lambda)$ -removable, and denote it by  $\{R_{pn}^\lambda\}$ . Lemma 3.1(iii) then says that

$$\lambda = \deg(R_{pn}^\lambda) - \deg(R_{p(n-1)}^\lambda) \quad \text{for every } n \geq 2, \quad (4)$$

*i.e.*, as we move from  $R_{p(n-1)}^\lambda$  to  $R_{pn}^\lambda$  in the sequence  $\{R_{pn}^\lambda\}$ , the degree of regularity always increases by  $\lambda$ .

Now we look at some small values of  $p$ .

**Example 5**  $p = 1$ , so  $\deg(R_{1n}^\lambda) = \lambda(n - 1)$  where  $\lambda = 0$  or  $1$ .

$\lambda = 0$  Here  $\deg(R_{1n}^0) = 0$ . So  $R_{1n}^0$  is the graph with  $n$  isolated vertices,  $nK_1$ . Clearly  $R_{11}^0 = K_1$  and, for  $n \geq 2$ , the removal of any  $K_1$  from  $nK_1$  gives the graph  $(n - 1)K_1$ , thus  $\{R_{1n}^0\} = \{nK_1\}$  is  $K_1$ -removable. It is also clear that  $\{nK_1\}$  is the unique regular  $(K_1, 0)$ -removable sequence.

$\lambda = 1$  Here  $\deg(R_{1n}^1) = n - 1$ .  $\{R_{1n}^1\} = \{K_n\}$  is the unique regular  $(K_1, 1)$ -removable sequence.

**Example 6**  $p = 2$ , so  $\deg(R_{2n}^\lambda) = \lambda(n - 1) + 1$  where  $\lambda = 0, 1$ , or  $2$ .

$\lambda = 0$  Here  $\deg(R_{2n}^0) = 1$ .  $\{R_{2n}^0\} = \{nK_2\}$  is the unique regular  $(K_2, 0)$ -removable sequence.

$\lambda = 1$  Here  $\deg(R_{2n}^1) = n$ . Now  $K_{n,n}$ , the complete bipartite graph on  $2n$  vertices, has degree  $n$ . Also  $K_{1,1} = K_2$  and, for every  $n \geq 2$ , the removal of any  $K_2$  from  $K_{n,n}$  results in  $K_{n-1,n-1}$ , *i.e.*  $K_{n,n} - K_2 = K_{n-1,n-1}$ . So one example of a regular  $(K_2, 1)$ -removable sequence is  $\{R_{2n}^1\} = \{K_{n,n}\}$ , we show in Corollary 3.2 that this is the unique example.

$\lambda = 2$  Here  $\deg(R_{2n}^2) = 2n - 1$ .  $\{R_{2n}^2\} = \{K_{2n}\}$  is the unique regular  $(K_2, 2)$ -removable sequence.

We see that the 1-st graph in each of the three sequences  $\{R_{2n}^0\} = \{nK_2\}$ ,  $\{R_{2n}^1\} = \{K_{n,n}\}$ , and  $\{R_{2n}^2\} = \{K_{2n}\}$  is  $K_2$ , *i.e.* we must think of  $K_2$  as being the three graphs  $1K_2 = K_{1,1} = K_{2,1}$ . Then the 2-nd graph in each sequence is obtained by changing the 1's in the notation for these graphs into 2's, etc. This is illustrated in Fig. 2, where the first 3 graphs in each of the three sequences  $\{nK_2\}$ ,  $\{K_{n,n}\}$ , and  $\{K_{2n}\}$  are shown. Note also that the  $n$ -th graphs in each of the three sequences are well-known graphs on  $2n$  vertices:  $nK_2$ ,  $K_{n,n}$ , and  $K_{2n}$ , respectively.

Recall that a  $K_p$ -removable sequence  $\{G_{pn}\}$  is regular if all of its graphs are regular, and is irregular if at least one of its graphs is irregular. The next result shows that all  $K_p$ -removable sequences for  $p = 1$  or  $p = 2$  are regular (and are those given in Examples 5 and 6). From Example 3 we see that irregular  $K_p$ -removable sequences exist for every  $p \geq 3$ .

**Corollary 3.2** *For  $p = 1$  or  $p = 2$  all  $K_p$ -removable sequences are regular, (and are given in Examples 5 and 6).*

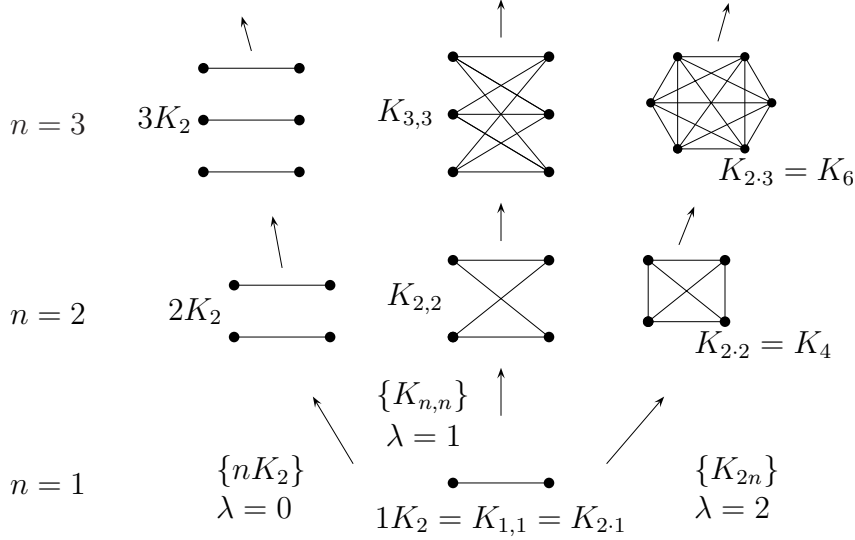


Figure 2: The first 3 graphs in each of the  $K_2$ -removable sequences  $\{nK_2\}$ ,  $\{K_{n,n}\}$ , and  $\{K_{2n}\}$ . These are the unique regular  $(K_2, 0)$ -,  $(K_2, 1)$ -, and  $(K_2, 2)$ -removable sequences, respectively. See Example 6.

*Proof.* ( $p = 2$ ) Let  $\{G_{2n}\}$  be a  $K_2$ -removable sequence, then  $G_{21} = K_2$ . A candidate for  $G_{22}$  must be a graph on 4 vertices with at least two disjoint  $K_2$ 's, and the removal of any  $K_2$  must leave a  $K_2$ . The only possibilities are: (a)  $2K_2$ , (b)  $K_{2,2}$ , or (c)  $K_{2,2} = K_4$ , all of which are regular.

(a) We prove by induction on  $n$  that if  $\{G_{2n}\}$  is an arbitrary  $K_2$ -removable sequence which begins  $\{K_2, 2K_2, \dots\}$  then it is the regular sequence  $\{nK_2\}$ .

Let  $n \geq 3$  and suppose that  $G_{2(n-1)} = (n-1)K_2$ . Now consider  $G_{2n}$  which is the union of  $(n-1)K_2$  and a 'new'  $K_2$  (and some edges between them); let edge  $(u, v)$  be this new  $K_2$ . If  $(u, v)$  is an isolated  $K_2$  then we are finished, so, assume that  $u$  is adjacent to vertex  $u_1$  in  $G_{2(n-1)}$ . Now  $G_{2(n-1)} = (n-1)K_2$

and  $n \geq 3$  so  $G_{2(n-1)}$  contains some edge  $(u_2, u_3)$  with  $u_2 \neq u_1$  and  $u_3 \neq u_1$ . But  $G_{2n} - (u_2, u_3)$  contains the path on the three vertices  $u_1, u$ , and  $v$  (or triangle if  $(v, u_1)$  is also an edge); a contradiction because  $G_{2n} - (u_2, u_3) = (n-1)K_2$ . Thus  $(u, v)$  must be an isolated  $K_2$  and the induction goes through, *i.e.*,  $G_{2n} = nK_2$  and so  $\{G_{2n}\} = \{nK_2\}$ .

(b) Similarly we use induction to show that any  $K_2$ -removable sequence  $\{G_{2n}\}$  which begins  $\{K_2 = K_{1,1}, K_{2,2}, \dots\}$  is the regular sequence  $\{K_{n,n}\}$ .

Let  $n \geq 3$  and suppose that  $G_{2(n-1)} = K_{n-1, n-1}$ ; let the 2 independent sets of  $G_{2(n-1)}$  be  $I_1 = \{u_1, \dots, u_{n-1}\}$  and  $I_2 = \{v_1, \dots, v_{n-1}\}$ , and let the new  $K_2$  of  $G_{2n}$  be the edge  $(u, v)$ . Now  $(u, v)$  cannot be isolated so let vertex  $u$  be adjacent to some vertices in  $G_{2(n-1)}$ . Suppose that  $u$  is adjacent to vertex  $u_{j_1} \in I_1$  and to vertex  $v_{j_2} \in I_2$ , then  $G_{2n}[u, u_{j_1}, v_{j_2}] = C_3$ , an odd cycle. Now remove any  $K_2 = (u_{j_3}, v_{j_4})$  where  $j_3 \neq j_1$  and  $j_4 \neq j_2$ , then  $G_{2n} - (u_{j_3}, v_{j_4}) = K_{n-1, n-1}$  still contains this odd cycle, a contradiction. Thus  $u$  is adjacent to vertices from exactly one of  $I_1$  or  $I_2$ , say  $I_2$ .

Now suppose that  $u$  is not adjacent to every vertex in  $I_2$ , suppose that it is not adjacent to  $v_{j_5}$ , say. Consider the edge  $(u_1, v_{j_6})$  where  $j_6 \neq j_5$ . In the graph  $G_{2n} - (u_1, v_{j_6}) = K_{n-1, n-1}$  the set of vertices  $\{u_2, \dots, u_{n-1}, u\} = \{I_1 \cup \{u\}\} \setminus \{u_1\}$  is one of the independent sets and  $\{I_2 \cup \{v\}\} \setminus \{v_{j_6}\}$  is the other. Thus  $(u, v_{j_5})$  is an edge, a contradiction. So  $u$  is adjacent to all vertices in  $I_2$ . Similarly  $v$  is adjacent to all vertices in  $I_1$ . Thus  $G_{2n} = K_{n,n}$  and  $\{G_{2n}\} = \{K_{n,n}\}$ .

The proofs for (c)  $K_{2,2} = K_4$  and the case  $p = 1$  are similar. ■

Before the next main result, Theorem 3.4, we need:

**Lemma 3.3** *For a fixed  $p \geq 1$  let  $\{R_{pn}^\lambda\}$  be regular  $(K_p, \lambda)$ -removable. Then, for every  $n \geq 1$ ,*

- (i)  $R_{pn}^\lambda$  contains  $\geq n$  disjoint  $K_p$ 's,
- (ii)  $R_{pn}^\lambda$  contains no  $K_{p+1}$ 's for  $0 \leq \lambda \leq p-1$ .

*Proof.* (i) See the first paragraph of Section 2.

(ii) Let  $0 \leq \lambda \leq p - 1$ . Suppose that for some  $p + 1$  vertices  $\{v_1, \dots, v_{p+1}\}$  of  $R_{pn}^\lambda$  we have  $R_{pn}^\lambda[v_1, \dots, v_{p+1}] = K_{p+1}$ . Consider  $R_{pn}^\lambda - K_p$  where  $K_p = R_{pn}^\lambda[v_1, \dots, v_p]$ . Clearly  $v_{p+1} \in R_{pn}^\lambda - K_p$  and  $\deg(v_{p+1})$  has decreased by  $p$ ; but from (4) it should have decreased by  $\lambda$ , a contradiction because  $0 \leq \lambda \leq p - 1$ . ■

Recall that  $K_{p \times n} = K_{\underbrace{n, \dots, n}_p}$  denotes the complete  $p$ -partite graph on  $pn$  vertices.

**Theorem 3.4** *For a fixed  $p \geq 2$  there is a unique regular  $(K_p, \lambda)$ -removable sequence for  $\lambda = 0, p - 1$ , or  $p$ :*

- (i)  $\{nK_p\}$  is the unique regular  $(K_p, 0)$ -removable sequence,
- (ii)  $\{K_{p \times n}\}$  is the unique regular  $(K_p, p - 1)$ -removable sequence,
- (iii)  $\{K_{pn}\}$  is the unique regular  $(K_p, p)$ -removable sequence.

*Proof.* The proofs of (i) and (iii) are straightforward.

(ii) ( $\lambda = p - 1$ ) We use induction on  $n$ . Let  $\{R_{pn}^{p-1}\}$  be an arbitrary regular  $(K_p, p - 1)$ -removable sequence; for  $n = 1$  we know that  $R_{p1}^{p-1} = K_p = K_{p \times 1}$ . Let  $n \geq 2$ , then  $R_{pn}^{p-1}$  contains a  $K_p = R_{pn}^{p-1}[v_1, \dots, v_p]$ , say. By the induction hypothesis we have  $R_{pn}^{p-1} - K_p = R_{p(n-1)}^{p-1} = K_{p \times (n-1)}$ . Let  $I_1, \dots, I_p$  be the  $p$  independent sets of  $R_{p(n-1)}^{p-1}$ , each has  $n - 1$  vertices. Now suppose that in  $R_{pn}^{p-1}$  a vertex from  $\{v_1, \dots, v_p\}$ , say  $v_i$ , is adjacent to a vertex from each of these  $p$  independent sets; call these vertices  $\{u_1, \dots, u_p\}$ , where  $u_k \in I_k$  for  $k = 1, \dots, p$ . Then  $R_{pn}^{p-1}[v_i, u_1, \dots, u_p] = K_{p+1}$ , a contradiction to Lemma 3.3(ii). Thus  $v_i$  is adjacent to vertices in at most  $p - 1$  of the independent sets  $I_1, \dots, I_p$ .

Now  $v_i$  is adjacent to  $(p - 1)(n - 1)$  vertices from  $I_1 \cup \dots \cup I_p$ , so it must be adjacent to all  $n - 1$  vertices in some  $p - 1$  of  $I_1, \dots, I_p$ . Suppose that in  $R_{pn}^{p-1}$  two distinct vertices from  $\{v_1, \dots, v_p\}$ , say  $v_{i_1}$  and  $v_{i_2}$ , are adjacent to all of the vertices in the same

$p - 1$  independent sets, say  $I_1, \dots, I_{p-1}$ . Let  $u_k \in I_k$  for  $k = 1, \dots, p - 1$ . Then  $R_{pn}^{p-1}[v_{i_1}, v_{i_2}, u_1, \dots, u_{p-1}] = K_{p+1}$ , again a contradiction. Thus, any two distinct vertices from  $\{v_1, \dots, v_p\}$  are adjacent to all vertices in distinct  $(p - 1)$ -sets of  $\{I_1, \dots, I_p\}$ .

For  $i = 1, \dots, p$  let  $v_i$  be non-adjacent to all vertices in the independent set  $I_{i'}$  for some  $i' = 1, \dots, p$ . By above the mapping  $v_i \leftrightarrow I_{i'}$  is a bijection. Then  $\{v_i\} \cup I_{i'}$  is an independent set in  $R_{pn}^{p-1}$ , and  $v_i$  is adjacent to all the vertices in all other independent sets. This is true for every  $i = 1, \dots, p$ . It is now clear that  $R_{pn}^{p-1} = K_{p \times n}$ , and so the induction goes through and  $\{R_{pn}^{p-1}\} = \{K_{p \times n}\}$ . ■

Note that this theorem connects three well-known graphs on  $pn$  vertices:  $nK_p$ ,  $K_{p \times n}$ , and  $K_{pn}$ . The case  $p = 2$  was discussed in Example 6 and illustrated in Fig. 2. Note also that the three regular sequences (i)  $\{nK_p\}$ , (ii)  $\{K_{p \times n}\}$ , and (iii)  $\{K_{pn}\}$  are the sequences  $\{H_{pn}^\Lambda\}$  where (i)  $\Lambda = (\underbrace{\emptyset, \dots, \emptyset}_p)$ , (ii)  $\Lambda = ([p] \setminus \{1\}, \dots, [p] \setminus \{p\})$ , and (iii)  $\Lambda = ([p], \dots, [p])$ , respectively. (In (iii) only  $\Lambda$  is not uniform.)

So, for  $p = 1$  or  $p = 2$  and all values of  $\lambda$ , and for every  $p \geq 3$  and  $\lambda = 0, p - 1$ , or  $p$ , there is a unique regular  $(K_p, \lambda)$ -removable sequence.

## 4 Levelled Cayley Graphs, more regular $(K_p, \lambda)$ -removable sequences

Generally our construction of  $\{H_{pn}^\Lambda\}$  from Section 2 gives irregular sequences. For an arbitrary vertex  $(i, j) \in H_{pn}^\Lambda$  we have  $\deg((i, j)) = \lambda_i(n - 1) + p - 1$ , see (2). So  $H_{pn}^\Lambda$  is regular if and only if  $\lambda_i$  is constant for all  $i \in [p]$ . We set  $\lambda_i = \lambda$  then  $H_{pn}^\Lambda$  is regular of degree  $\lambda(n - 1) + p - 1$ , (see Lemma 3.1(ii)), and  $\{H_{pn}^\Lambda\}$  is a regular  $(K_p, \lambda)$ -removable sequence.

We now construct a regular  $(K_p, \lambda)$ -removable sequence us-

ing a finite group. This combines the construction of  $\{H_{pn}^\Lambda\}$  from Section 2 with the construction of a Cayley graph; see, for example, p. 122 of Biggs [2].

Let  $p \geq 3$  and let  $\mathcal{G}_p = \{g_1, \dots, g_p\}$  be a finite group with  $p$  elements, where  $g_1 = e$  is the identity element. Let  $\Lambda \subseteq \mathcal{G}_p$  be a subset of  $\mathcal{G}_p$  with  $e \notin \Lambda$  and  $|\Lambda| = \lambda$ .

Now consider the following graph, a *levelled Cayley* graph,  $\Gamma_n = \Gamma_n(\mathcal{G}_p, \Lambda)$ : It has  $n \geq 1$  levels of vertices, each level having  $p$  vertices. For any fixed  $j$  with  $1 \leq j \leq n$  the vertices at level  $j$  are  $\{(g_1, j), \dots, (g_p, j)\} = \{(g, j) \mid g \in \mathcal{G}_p\}$ , and the edges are of two types:

(i) *fixed-level* edges, say at level  $j$

$$((g, j), (h, j)) \text{ is an edge for all } g, h \in \mathcal{G}_p \text{ where } g \neq h;$$

(i.e., the ‘fixed-level’ graph is  $K_p$ ), and

(ii) *cross-level* edges, for  $j > j'$

$$((g, j), (g', j')) \text{ is an edge if and only if } g'g^{-1} \in \Lambda.$$

Using Theorem 2.3 it is straightforward to prove:

**Theorem 4.1** *For any finite group  $\mathcal{G}_p$  with  $p$  elements and any  $\Lambda \subseteq \mathcal{G}_p$  with  $e \notin \Lambda$  and  $|\Lambda| = \lambda$ , the sequence  $\{\Gamma_n(\mathcal{G}_p, \Lambda)\}$  is regular  $(K_p, \lambda)$ -removable. ■*

Now, for any  $\mathcal{G}_p$  with  $p \geq 3$  and for each  $\lambda = 0, \dots, p-1$ , there exists a  $\Lambda$  satisfying the requirements in Theorem 4.1, and Theorem 3.4(iii) takes care of  $\lambda = p$ , so we have the following existence result for regular  $(K_p, \lambda)$ -removable sequences.

**Theorem 4.2** *For any  $p \geq 3$  and any  $\lambda = 0, \dots, p$ , there exists a regular  $(K_p, \lambda)$ -removable sequence, namely the sequence  $\{\Gamma_n(\mathcal{G}_p, \Lambda)\}$ , where  $\mathcal{G}_p$  is any finite group of order  $p$  and  $\Lambda$  is any subset of  $\mathcal{G}_p$  with  $e \notin \Lambda$  and  $|\Lambda| = \lambda$ . ■*

Let  $\bar{\Lambda}$  denote the complement of  $\Lambda$  in  $\mathcal{G}_p$  and let  $\langle \bar{\Lambda} \rangle$  be the subgroup generated by  $\bar{\Lambda}$ , also let  $\langle \bar{\Lambda} \rangle g$  denote a typical coset of this subgroup.

Let  $V = \Gamma_n[(g_1, v_1), \dots, (g_p, v_p)]$  be an arbitrary induced subgraph in  $\Gamma_n(\mathcal{G}_p, \Lambda)$  with exactly one vertex from each independent set  $I_i = \{(g_i, j) \mid 1 \leq j \leq n\}$ , where  $i = 1, \dots, p$ . We are interested in the  $K_p$ 's in  $\Gamma_n(\mathcal{G}_p, \Lambda)$ . The next theorem gives a necessary and sufficient condition for this  $V$  to be a  $K_p$ , this condition is cleaner than the corresponding condition of Theorem 2.4 for the graph  $H_{pn}^\Lambda$ ; it also enables us to count the number of  $K_p$ 's in  $\Gamma_n(\mathcal{G}_p, \Lambda)$ .

As in Section 2 let  $V$  have vertices at  $m$  different levels:  $\ell_1 < \dots < \ell_m$ . For  $1 \leq k \leq m$ , let  $V_k = \{g_i \mid v_i = \ell_k\} \neq \emptyset$  be the set of first coordinates of all vertices of  $V$  at level  $\ell_k$ . Then the sets  $V_1, \dots, V_m$  partition  $\mathcal{G}_p$ , and we have:

**Theorem 4.3** *With the above notation  $V = K_p$  if and only if for every  $k$ , with  $1 \leq k \leq m$ ,  $V_k$  is a union of cosets of  $\langle \bar{\Lambda} \rangle$ .*

*Proof.* Suppose that  $V = \Gamma_n[(g_1, v_1), \dots, (g_p, v_p)] = K_p$ . Now  $\ell_m = \max\{v_1, \dots, v_p\}$  is the highest level of  $\Gamma_n(\mathcal{G}_p, \Lambda)$  which contains vertices from  $V$ , and  $V_m = \{g_i \mid v_i = \ell_m\}$  is the set of first coordinates of all vertices of  $V$  at this highest level.

Let  $s$  be an arbitrary element in  $V_m \subseteq \mathcal{G}_p$ . Now consider the graph  $V' = \Gamma_n[(g_1 s^{-1}, v_1), \dots, (g_p s^{-1}, v_p)]$ , let us reorder the vertices in  $V'$  so that  $V' = \Gamma_n[(g_1, v'_1), \dots, (g_p, v'_p)]$ ; we use this version of  $V'$ . The graph  $V'$  is also a  $K_p$  and the highest level of its vertices is  $\ell_m$  also. Then  $V'_m = \{g_i \mid v'_i = \ell_m\} = V_m s^{-1}$  is the set of first coordinates of these vertices. We show that  $V'_m$  is a union of cosets of  $\langle \bar{\Lambda} \rangle$ .

First we show that  $\bar{\Lambda} \subseteq V'_m$ . Now  $g_1 = e \in \bar{\Lambda}$  and  $s \in V_m$ , so certainly  $e = s s^{-1} \in V'_m$ . So vertex  $(g_1, v'_1) = (e, \ell_m) \in V'$  lies at level  $\ell_m$ . Let  $g_i \neq e$  and let  $g_i \in \bar{\Lambda}$ , but suppose that  $g_i \notin V'_m$ . This means that vertex  $(g_i, v'_i) \in V'$  does not lie at level  $\ell_m$  so it must lie at a lower level, *i.e.*,  $\ell_m > v'_i$ . Now, because

$((e, \ell_m), (g_i, v'_i))$  is an edge in  $V'$  then  $((s, \ell_m), (g_i s, v'_i))$  is an edge in  $V$  and so in  $\Gamma_n(\mathcal{G}_p, \Lambda)$ ; so  $g_i s s^{-1} = g_i \in \Lambda$ , a contradiction because  $g_i \in \overline{\Lambda}$ . Thus for every  $g_i \in \overline{\Lambda}$ , then  $g_i \in V'_m$ , *i.e.*,  $\overline{\Lambda} \subseteq V'_m$ .

Next we show that  $\langle \overline{\Lambda} \rangle \subseteq V'_m$ . For any  $r \geq 1$  let  $\Pi(r)$  denote a product of  $r$  arbitrary elements from  $\overline{\Lambda}$ . By induction we show, for any fixed  $r \geq 1$ , that every  $\Pi(r) \in V'_m$ . Now  $\overline{\Lambda} \subseteq V'_m$ , *i.e.*, every  $\Pi(1) \in V'_m$ . Assume for some  $r$  with  $r \geq 1$  that every  $\Pi(r) \in V'_m$ . Now suppose that some  $\Pi(r+1)$ , say  $h(r+1) = a_1 \cdots a_{r+1} \notin V'_m$ , but each  $a_1, \dots, a_{r+1} \in \overline{\Lambda}$ . Let  $h(r) = a_2 \cdots a_{r+1}$ , then, by the induction hypothesis,  $h(r) \in V'_m$  so vertex  $(h(r), \ell_m) \in V'$  is at the highest level  $\ell_m$  and the vertex in  $V'$  with first coordinate  $h(r+1)$  is at a lower level. There is an edge between these two vertices so  $h(r+1)h(r)^{-1} = a_1 \in \Lambda$ , a contradiction because  $a_1 \in \overline{\Lambda}$ . Thus  $h(r+1) \in V'_m$  and the induction goes through, and so  $\langle \overline{\Lambda} \rangle \subseteq V'_m$ .

Finally we show that  $V'_m$  is a union of cosets of  $\langle \overline{\Lambda} \rangle$ . Let  $g \in V'_m$ , but suppose that  $g \notin \langle \overline{\Lambda} \rangle$ . Then, by similar reasoning to before, we see that every  $\Pi(1)g \in V'_m$ , and by induction, that every  $\Pi(r)g \in V'_m$  for every  $r \geq 1$ ; thus the coset  $\langle \overline{\Lambda} \rangle g \subseteq V'_m$ . Thus  $V'_m$  is a union of cosets of  $\langle \overline{\Lambda} \rangle$ , and so  $V_m = V'_m s$  is also.

Now we return to the graph  $V$  and show that  $V_k$  is a union of cosets of  $\langle \overline{\Lambda} \rangle$  for every  $1 \leq k \leq m$ . From above this is true for  $V_m$ , so assume for all  $k$  with  $k \leq m$  that  $V_k$  is a union of cosets of  $\langle \overline{\Lambda} \rangle$ , we show that  $V_{k-1}$  is also.

Let  $g \in V_{k-1}$ , we show that the coset  $\langle \overline{\Lambda} \rangle g \subseteq V_{k-1}$ . As before first we show that every  $\Pi(1)g \in V_{k-1}$ , for, suppose not, then:

Either  $\Pi(1)g \in V_{k'}$  where  $k' > k-1$ , so, by the induction hypothesis,  $V_{k'}$  is a union of cosets of  $\langle \overline{\Lambda} \rangle$ . Thus  $\langle \overline{\Lambda} \rangle \Pi(1)g = \langle \overline{\Lambda} \rangle g \subseteq V_{k'}$ , so  $g \in V_{k'}$ , a contradiction because  $g \in V_{k-1}$ ;

Or  $\Pi(1)g \in V_{k'}$  where  $k' < k-1$ , *i.e.*, level  $\ell_{k'}$  is lower than level  $\ell_{k-1}$ . So vertex  $(\Pi(1)g, \ell_{k'}) \in V$  is at a lower level than vertex  $(g, \ell_{k-1}) \in V$ , so  $\Pi(1)g g^{-1} = \Pi(1) \in \Lambda$ , again a contradiction.

So  $\Pi(1)g \in V_{k-1}$ , and we proceed by induction as before to show

that for any  $r \geq 1$  then every  $\prod(r)g \in V_{k-1}$  and so  $\langle \overline{\Lambda} \rangle g \subseteq V_{k-1}$ , and the induction on  $k$  goes through; thus  $V_{k-1}$  is a union of cosets of  $\langle \overline{\Lambda} \rangle$ .

So, in conclusion, for any  $k$  with  $1 \leq k \leq m$ ,  $V_k$  is a union of cosets of  $\langle \overline{\Lambda} \rangle$ , as required.

For the converse let each  $V_k$  be a union of cosets of  $\langle \overline{\Lambda} \rangle$ . Let  $(g, \ell_k)$  and  $(g', \ell_{k'})$  be two arbitrary vertices in  $V$ , we show that  $((g, \ell_k), (g', \ell_{k'}))$  is an edge in  $\Gamma_n(\mathcal{G}_p, \Lambda)$ . If  $\ell_k = \ell_{k'}$  then, certainly,  $((g, \ell_k), (g', \ell_{k'}))$  is an edge by construction of  $\Gamma_n(\mathcal{G}_p, \Lambda)$ . Otherwise, without loss of generality, let  $\ell_k > \ell_{k'}$ . Then  $g$  and  $g'$  are in different cosets of  $\langle \overline{\Lambda} \rangle$ , so  $g'g^{-1} \notin \langle \overline{\Lambda} \rangle$ , so  $g'g^{-1} \in \overline{\langle \overline{\Lambda} \rangle} \subseteq \Lambda$ , and again  $((g, \ell_k), (g', \ell_{k'}))$  is an edge. Thus  $V = K_p$  as required. ■

Now we count the number of  $K_p$ 's in  $\Gamma_n(\mathcal{G}_p, \Lambda)$ ; let  $|\mathcal{G}_p : \langle \overline{\Lambda} \rangle|$  be the index of  $\langle \overline{\Lambda} \rangle$  in  $\mathcal{G}_p$ , i.e., the number of cosets of  $\langle \overline{\Lambda} \rangle$  in  $\mathcal{G}_p$ .

**Corollary 4.4** *The number of  $K_p$ 's in  $\Gamma_n(\mathcal{G}_p, \Lambda)$  is given by*

$$n^{|\mathcal{G}_p : \langle \overline{\Lambda} \rangle|}.$$

*Proof.* Consider any coset  $\langle \overline{\Lambda} \rangle g$ , let us ‘place’ the elements of this coset at any fixed level  $j$ , where  $1 \leq j \leq n$ , in the graph  $\Gamma_n(\mathcal{G}_p, \Lambda)$ . Each such placement of every coset of  $\langle \overline{\Lambda} \rangle$  gives a  $K_p$  and every  $K_p$  corresponds to such a placement of every coset of  $\langle \overline{\Lambda} \rangle$ . Hence, the number of  $K_p$ 's in  $\Gamma_n(\mathcal{G}_p, \Lambda)$  equals the number of such placements of all the cosets of  $\langle \overline{\Lambda} \rangle$ . There are  $|\mathcal{G}_p : \langle \overline{\Lambda} \rangle|$  cosets, and  $n$  levels to place each, hence  $n^{|\mathcal{G}_p : \langle \overline{\Lambda} \rangle|}$  such placements and so  $n^{|\mathcal{G}_p : \langle \overline{\Lambda} \rangle|}$  corresponding  $K_p$ 's. ■

**Example 7** (a) Let  $\text{Sym}(3) = \{e, (12), (13), (23), (123), (132)\}$  be the symmetric group on  $\{1, 2, 3\}$ , and let  $\Lambda = \{(12), (13), (23), (123)\}$ . Then  $\overline{\Lambda} = \{e, (132)\}$  and  $\langle \overline{\Lambda} \rangle = \{e, (123), (132)\}$ ; the other coset of  $\langle \overline{\Lambda} \rangle$  is  $\langle \overline{\Lambda} \rangle(12) = \{(12), (13), (23)\}$ . Consider the graph  $\Gamma_2(\text{Sym}(3), \Lambda)$  shown in Fig. 3(a), we have  $|\text{Sym}(3) :$

$|\langle \bar{\Lambda} \rangle| = 2$  so this graph has  $2^2 = 4$   $K_6$ 's. They are the 2 fixed-level  $K_6$ 's,  $\Gamma_2[(e, 1), ((123), 1), ((132), 1), ((12), 2), ((13), 2), ((23), 2)]$ , and  $\Gamma_2[(e, 2), ((123), 2), ((132), 2), ((12), 1), ((13), 1), ((23), 1)]$ . This is the 2-nd graph in the regular  $(K_6, 4)$ -removable sequence  $\{\Gamma_n(\text{Sym}(3), \Lambda)\}$ .

(b) Let  $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$  be the additive group (mod 6), and let  $\Lambda = \{1, 2, 4, 5\}$ . Then  $\Lambda = \langle \bar{\Lambda} \rangle = \{0, 3\}$  and the other cosets of  $\langle \bar{\Lambda} \rangle$  are  $\{1, 4\}$  and  $\{2, 5\}$ . Thus  $|\mathbb{Z}_6 : \langle \bar{\Lambda} \rangle| = 3$  and  $\Gamma_2(\mathbb{Z}_6, \Lambda)$  has  $2^3 = 8$   $K_6$ 's, see Fig. 3(b). One such  $K_6$  is  $\Gamma_2[(0, 2), (3, 2), (1, 1), (4, 1), (2, 1), (5, 1)]$ , and another is  $\Gamma_2[(0, 2), (3, 2), (1, 1), (4, 1), (2, 2), (5, 2)]$ , in which the vertices corresponding to coset  $\{2, 5\}$  have been moved up one level. This is the 2-nd graph in the regular  $(K_6, 4)$ -removable sequence  $\{\Gamma_n(\mathbb{Z}_6, \Lambda)\}$ .

Note that the two graphs in Example 7 are non-isomorphic because they have a different number of  $K_6$ 's, they are the 2-nd graphs in two different regular  $(K_6, 4)$ -removable sequences. So, in general, regular  $(K_p, p - 2)$ -removable sequences are not unique.

**Example 8** Consider the graphs  $\Gamma_3(\mathbb{Z}_4, \{1\})$  and  $\Gamma_3(\mathbb{Z}_4, \{2\})$ , shown in Figs. 4(a) and (b) respectively. Both graphs have 3  $K_4$ 's. In  $\Gamma_3(\mathbb{Z}_4, \{1\})$  the union of all edges which lie outside its 3  $K_4$ 's is a  $C_{12}$ , however in  $\Gamma_3(\mathbb{Z}_4, \{2\})$  this union is  $C_6 \cup C_6$ . Thus  $\Gamma_3(\mathbb{Z}_4, \{1\}) \neq \Gamma_3(\mathbb{Z}_4, \{2\})$  and so  $\{\Gamma_n(\mathbb{Z}_4, \{1\})\} \neq \{\Gamma_n(\mathbb{Z}_4, \{2\})\}$ , and we have two different regular  $(K_4, 1)$ -removable sequences.

Example 8 illustrates that, in general, regular  $(K_p, 1)$ -removable sequences are not unique.

Theorem 3.4 states that, for a fixed  $p \geq 2$ , the only regular  $(K_p, 0)$ -,  $(K_p, p - 1)$ -, and  $(K_p, p)$ -removable sequences are unique. Above we give two different examples of a regular  $(K_p, 1)$ -removable sequence, and two different examples of a regular  $(K_p, p - 2)$ -removable sequence; thus, in general, regular  $(K_p, \lambda)$ -removable sequences are not unique unless  $\lambda = 0, p - 1$ ,

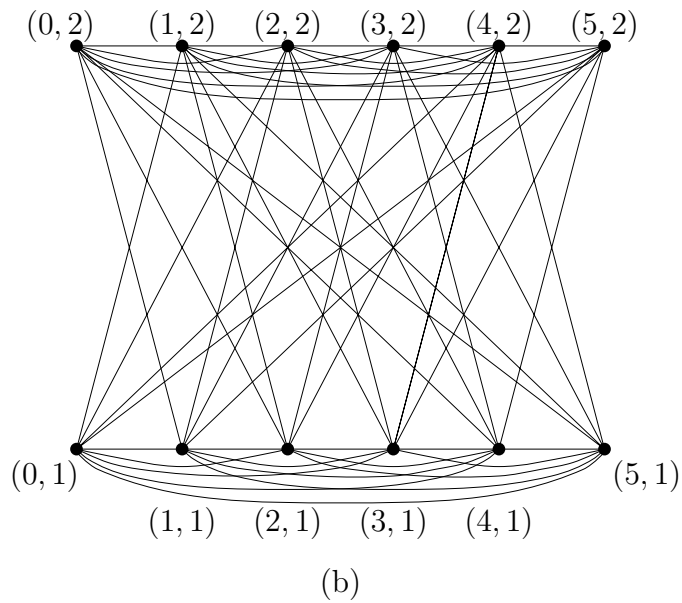
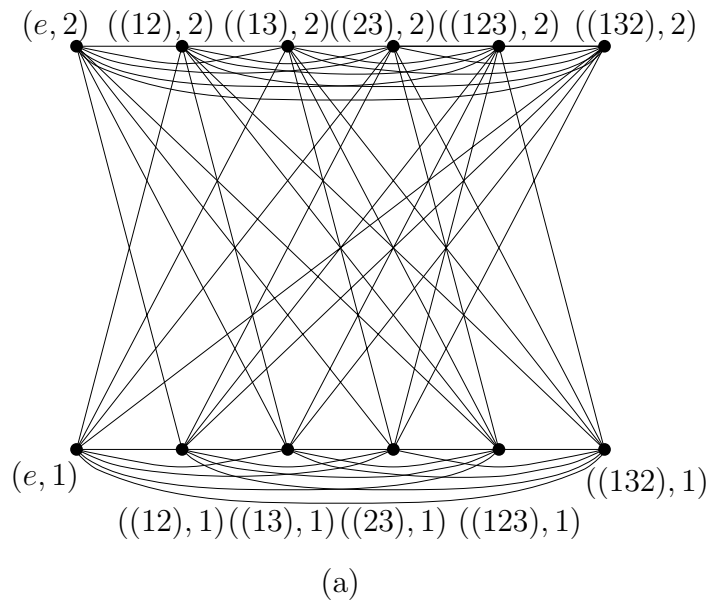


Figure 3: The non-isomorphic graphs (a)  $\Gamma_2(\text{Sym}(3), \Lambda)$  where  $\Lambda = \{(12), (13), (23), (123)\}$ , and (b)  $\Gamma_2(\mathbb{Z}_6, \Lambda)$  where  $\Lambda = \{1, 2, 4, 5\}$ . See Example 7.

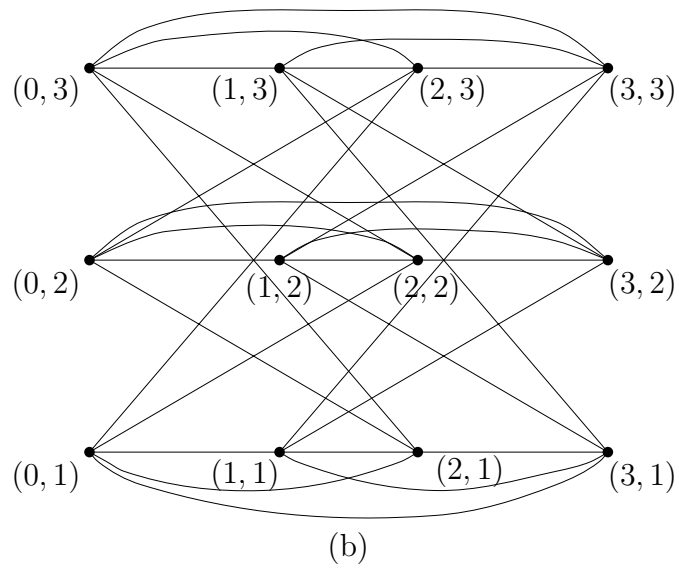
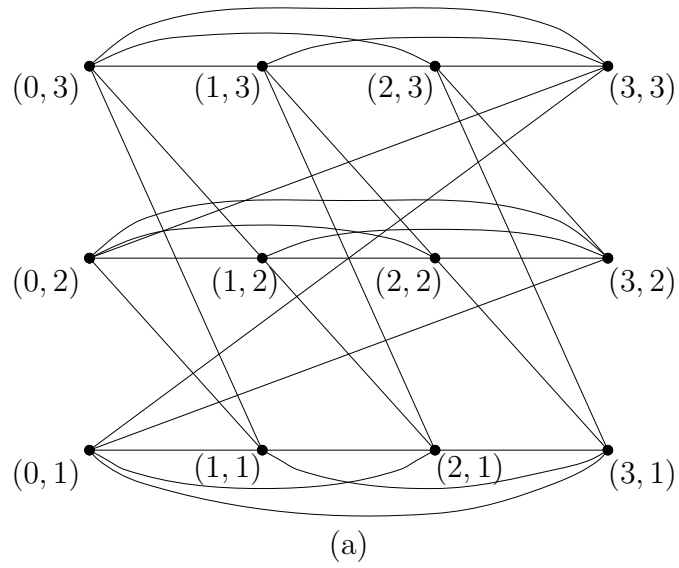


Figure 4: The non-isomorphic graphs (a)  $\Gamma_3(\mathbb{Z}_4, \{1\})$  and (b)  $\Gamma_3(\mathbb{Z}_4, \{2\})$ . See Example 8.

or  $p$ .

Some final comments:

If, for some  $\mathcal{G}_p$  and  $\Lambda$  we have  $|\mathcal{G}_p : \langle \overline{\Lambda} \rangle| = 1$ , then the graph  $\Gamma_n(\mathcal{G}_p, \Lambda)$  has exactly  $n$   $K_p$ 's which is the least number allowed by Lemma 3.3(i). Such a graph has no cross-level  $K_p$ 's; *e.g.*,  $\Gamma_3(\mathbb{Z}_4, \{1\})$  shown in Fig. 4(a).

It is also worth mentioning that for any group  $\mathcal{G}_p$ : (see Theorem 3.4)

- (i)  $\{\Gamma_n(\mathcal{G}_p, \emptyset)\} = \{nK_p\}$  is the unique regular  $(K_p, 0)$ -removable sequence,
- (ii)  $\{\Gamma_n(\mathcal{G}_p, \mathcal{G}_p \setminus \{e\})\} = \{K_{p \times n}\}$  is the unique regular  $(K_p, p-1)$ -removable sequence,
- (iii)  $\{\Gamma_n(\mathcal{G}_p, \mathcal{G}_p)\} = \{K_{pn}\}$  is the unique regular  $(K_p, p)$ -removable sequence.

(Note that in (iii) we do not have  $e \notin \Lambda$ .) Thus the unique regular  $(K_p, \lambda)$ -removable sequences for  $\lambda = 0, p-1$ , or  $p$  can all be constructed from an arbitrary group  $\mathcal{G}_p$ .

For other papers on graph sequences see Barefoot, Entringer, and Jackson [1], and the references therein; another somewhat related paper is Duchet, Tuza, and Vestergaard [3].

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## References

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