On the Symmetry of Energy Minimising Deformations in Nonlinear Elasticity II: Compressible Materials

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Abstract. Consider a homogeneous, isotropic, hyperelastic body occupying the annular region $A = \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\}$ in its reference state and subject to radially symmetric displacement, or mixed displacement/traction, boundary conditions. In Part I it was shown that if the body is composed of an incompressible material, then to each isochoric deformation of A there corresponds a radial isochoric deformation that has less elastic energy than the given deformation, provided that the stored-energy function is polyconvex and grows sufficiently rapidly at infinity. In this paper that analysis is further developed and extended to the compressible case for a large class of polyconvex constitutive relations.

The key ingredient is a new radial-symmetrisation procedure that is appropriate for problems where the symmetrised mapping must be one-to-one in order to prevent interpenetration of matter. For the pure displacement boundary-value problem, the radial symmetrisation of an orientation preserving diffeomorphism $\mathbf{u} : A \to A^*$ between annuli A and A^* is the deformation

$$\mathbf{u}^{\mathrm{rad}}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \qquad R = |\mathbf{x}|,$$

that maps each sphere $S_R \subset A$, of radius $R > 0$, centred at the origin into another such sphere $S_r = \mathbf{u}^{\mathrm{rad}}(S_R) \subset A^*$ that encloses the same volume as $\mathbf{u}(S_R)$. Since the volumes enclosed by the surfaces $\mathbf{u}(S_R)$ and $\mathbf{u}^{\mathrm{rad}}(S_R)$ are equal, the classical isoperimetric inequality then implies that $\mathrm{Area}(\mathbf{u}^{\mathrm{rad}}(S_R)) \leq \mathrm{Area}(\mathbf{u}(S_R))$. The equality of the enclosed volumes together with this reduction in surface area is then shown to give rise to a reduction in total energy for many of the constitutive relations used in nonlinear elasticity.

These results are also extended to classes of Sobolev deformations and applied to prove that the radially symmetric solutions to these boundary-value problems are local or global energy minimisers in various classes of (possibly non-symmetric) deformations of the annulus.

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