A Product Property of Sobolev Spaces with Application to Elliptic Estimates

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Abstract. In this paper a Sobolev inequality, which generalizes the Banach algebra property of such spaces, is established; For $p \in [1,\infty)$, $n, m \in \mathbb{Z}^+$, and $m \geq 2$ that satisfy $m > n/p$,

$$\|\phi\psi\|_{m,p,\Omega} \leq K \left[\left(\sup_{\Omega_s} |\phi| \right) \|\psi\|_{m,p,\Omega} + \Big(\|\psi\|_{m-1,q,\Omega} + \|\psi\|_{m-1,p,\Omega} \Big) \|\phi\|_{m,p,\Omega} \right]$$

for all $\phi, \psi \in W^{m,p}(\Omega)$ that satisfy $\operatorname{spt}\psi \subset \Omega_s \subset \Omega$ and domains $\Omega \subset \mathbb{R}^n$ that are nonempty, open, and satisfy the cone condition. Here $q = p$ if $p > n$, $q \in (n/\Upsilon, pn/(n-p)]$ if $n > p$, $q \in (n/\Upsilon, \infty)$ if $p = n$, $K = K(n, p, m, q, \mathcal{C})$, where $\mathcal{C}$ is the cone from the cone condition, and $\Upsilon := [\![n/p]\!]$, the largest integer less than or equal to n/p.

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1 Introduction; Sobolev Spaces

A standard classical methodology used to obtain a priori estimates for elliptic systems of partial differential equations is to first prove the required estimate when the system has constant coefficients and the region has smooth boundary and then use a partition of unity to extend the estimate to coefficients that depend on position and regions that are less regular. For example, Agmon, Douglis, and Nirenberg [3, 4] first establish the estimate (in the notation from Elasticity)

$$\|\mathbf{u}\|_{m+1,p,\Omega} \le N\left[\left\|\operatorname{Div}\mathsf{C}[\nabla\mathbf{u}]\right\|_{m-1,p,\Omega} + \left\|\mathsf{C}[\nabla\mathbf{u}]\mathbf{n}\right\|_{m-\frac{1}{p},p,\mathcal{S}} + \|\mathbf{u}\|_{p,\Omega}\right] \tag{1.1}$$

for all $\mathbf{u} \in W^{m+1,p}(\Omega;\mathbb{R}^n)$ that satisfy $\mathbf{u} = \mathbf{0}$ on $\mathcal{D}$, when $\mathsf{C} : \mathrm{M}^{n\times n} \to \mathrm{M}^{n\times n}$ is a constant linear mapping of the $n \times n$ matrices $\mathrm{M}^{n\times n}$ and Ω is a ball with $\mathcal{D} = \partial\Omega$ and also when Ω is a half-ball with either $\mathcal{D} = \partial\Omega$ or $\mathcal{D}$ is the curved portion of the boundary of the half-ball. Here $m \in \mathbb{Z}^+$, $p \in (1,\infty)$, $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$,

$$\|\mathbf{u}\|_{p,\Omega}^p := \int_\Omega |\mathbf{u}(\mathbf{x})|^p \,\mathrm{d}\mathbf{x}, \qquad (\nabla\mathbf{u})_i := \frac{\partial\mathbf{u}}{\partial x_i},$$

$$\|\mathbf{u}\|_{m,p,\Omega}^p := \sum_{|\boldsymbol{\alpha}|\le m} \|D^{\boldsymbol{\alpha}}\mathbf{u}\|_{p,\Omega}^p\,, \qquad (\operatorname{Div}\mathbf{M})_i := \sum_{j=1}^n \frac{\partial M_{ij}}{\partial x_j},$$

and $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n)$ is a multi-index with $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$ and $D^{\boldsymbol{\alpha}} = \partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}$.

For a general bounded open region $\Omega \subset \mathbb{R}^n$, a suitable open covering of Ω and $\partial\Omega$, by balls and half-balls, together with a partition of unity can then be used (see [3]) to prove (1.1) for $\mathsf{C}(\mathbf{x}) : \mathrm{M}^{n\times n} \to \mathrm{M}^{n\times n}$ whose components are m-times continuously differentiable on $\overline{\Omega}$. More generally, if one wants to establish[1] (1.1) for $\mathsf{C} \in W^{m,p}(\Omega)$, then one can make use of Moser's [7, pp. 273–274] tame inequality (see Klainerman and Majda [6, pp. 516–517] for a nice proof): If $1 \le p < \infty$ and $k \in \mathbb{Z}^+$, then there exists a constant $C = C(n,p,k) > 0$ such that

$$C^{-1}\|\phi\psi\|_{k,p,\mathbb{R}^n} \le \|\phi\|_{\infty,\mathbb{R}^n}\|\psi\|_{k,p,\mathbb{R}^n} + \|\psi\|_{\infty,\mathbb{R}^n}\|\phi\|_{k,p,\mathbb{R}^n} \tag{1.2}$$

for all $\phi,\psi \in W^{k,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

However, (1.2) is an inequality for Sobolev functions defined on all of $\mathbb{R}^n$, while in practice one must make use of this inequality for Sobolev functions defined on a bounded open region $\Omega \subset \mathbb{R}^n$. This presents no difficulties when the boundary of Ω is sufficiently smooth since one can then use standard extension results to obtain a version of (1.2) for such domains. When the boundary of the region is not smooth there are some unresolved difficulties.[2]

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[1] See, e.g., [11] for a proof of (1.1) when the components of C are Sobolev functions.

[2] See Maz′ya and Shaposhnikova [8, Chapter 7] for regions whose boundary can be parametrized by an appropriate Sobolev function. See, also, Nečas [10].

The main purpose of this paper is to provide a partial resolution of these difficulties by proving an inequality, which is similar to (1.2) and which is also useful for elliptic estimates, for regions that satisfy (only) a cone condition. In particular we show that if $\Omega \subset \mathbb{R}^n$ is nonempty, open, and satisfies the cone condition and if $p \in [1,\infty)$, $n, m \in \mathbb{Z}^+$, $m \geq 2$, and $p \in (n/m, n)$, then for any $q \in (n/[\![\frac{n}{p}]\!], pn/(n-p)]$ there is a constant $K = K(n,p,m,q,\mathcal{C})$ such that

$$\|\phi_1\phi_2\|_{m,p,\Omega} \leq K\left[\left(\sup_{\Omega_s}|\phi_1|\right)\|\phi_2\|_{m,p,\Omega} + \left(\|\phi_2\|_{m-1,q,\Omega} + \|\phi_2\|_{m-1,p,\Omega}\right)\|\phi_1\|_{m,p,\Omega}\right] \tag{1.3}$$

for all $\phi_1, \phi_2 \in W^{m,p}(\Omega)$ that satisfy $\operatorname{spt}\phi_2 \subset \Omega_s \subset \Omega$. Here $\mathcal{C}$ is the cone from the cone condition[3] and $[\![x]\!]$ is the largest integer less than or equal to x.

We note that our inequality, unlike (1.2), has the interesting feature that its dependence on the supremum of the first function is limited to the region that supports the second function. Our initial motivation for studying such inequalities originated in problems of global bifurcation[4] for the strongly-elliptic system that governs the equilibrium of elastic materials. In this context inequality (1.3) extends results of Valent [12, pp. 22-27] that are used to improve estimates in [3, 4] in order to apply them to elasticity. Our proof makes use of the following special cases of the usual Sobolev inequalities.

Proposition (See, e.g., [2, pp. 85–86])**.** *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open region that satisfies the cone condition. Suppose $1 \leq p < \infty$, $k \in \mathbb{Z}^+$, and $j \in \mathbb{N}$. Define $p_k := pn/(n-kp)$ if $n > kp$ and $p_k := \infty$ otherwise. Then there exist a constant $K = K(n,p,k,j,q,\mathcal{C})$, where $\mathcal{C}$ is the cone from the cone condition, that has the following properties.*

I. (Sobolev Imbedding Theorem) *If $k > n/p$ then $W^{k,p}(\Omega) \subset C_B(\Omega)$ with*

$$\sup_{\Omega}|\phi| \leq K\|\phi\|_{k,p,\Omega} \text{ for all } \phi \in W^{k,p}(\Omega).$$

II. (Gagliardo-Nirenberg-Sobolev Inequality) *If $k \leq n/p$ then $W^{k+j,p}(\Omega) \subset W^{j,q}(\Omega)$ with*

$$\|\phi\|_{j,q,\Omega} \leq K\|\phi\|_{k+j,p,\Omega} \text{ for all } \phi \in W^{k+j,p}(\Omega)$$

and $q \in [p, p_k]$ if $p_k < \infty$ and $q \in [p,\infty)$ otherwise.

Here,

$$C_B(\Omega) := \{\phi \in C(\Omega) : \phi \in L^\infty(\Omega)\},$$

which is a Banach space under the L^∞-norm.

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[3] I.e., every $\mathbf{x} \in \Omega$ is the vertex of cone that is contained in Ω and is a rigid deformation of $\mathcal{C}$. See, e.g., [1, p. 66] or [2, p. 82].

[4] See, e.g., [5, 9].

2 The Product Property

For a Sobolev function $\psi \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$, we define *the support of* ψ by

$$\operatorname{spt}\psi := \Omega \setminus \{\mathbf{x} \in \Omega : \psi(\mathbf{z}) = 0 \text{ for a.e. } \mathbf{z} \text{ in some open neighborhood of } \mathbf{x}\}.$$

Thus, since the complement of $\operatorname{spt}\psi$ is an open set, $D^\alpha\psi = 0$ a.e. on $\Omega \backslash (\operatorname{spt}\psi)$ for $|\alpha| \le k$. Consequently, if $\phi, \psi \in W^{k,p}(\Omega)$, $k > n/p$, and $\operatorname{spt}\psi \subset \Omega_s$ then $\phi(D^\gamma\psi) \in L^p(\Omega)$ for $|\gamma| \le k$ and

$$\left\|\phi(D^\gamma\psi)\right\|_{p,\Omega} \le \left(\sup_{\Omega_s}|\phi|\right)\left\|D^\gamma\psi\right\|_{p,\Omega}. \tag{2.1}$$

The main result of this paper is the following theorem, which generalizes the usual Banach algebra property of $W^{m,p}$, $m > n/p$. See Valent [12, pp. 22–27] for similar results.

Theorem. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$, be a nonempty open region that satisfies the cone condition. Suppose $1 \le p < \infty$, $m \in \mathbb{Z}^+$ with $m > n/p$, and $\Omega_s \subset \Omega$ is measurable. Then for every $q \in (\frac{n}{\Upsilon}, np/(n-p)]$, if $n > p$, and for every $q \in (\frac{n}{\Upsilon}, \infty)$, if $p = n$, there exists a constant $K = K(n,p,m,q,\mathcal{C}) > 0$, where $\mathcal{C}$ is the cone from the cone condition for Ω and $\Upsilon := [\![\frac{n}{p}]\!]$, such that the following are satisfied.*

(a) *If $m = 1$ then for all $\phi, \psi \in W^{1,p}(\Omega)$ that satisfy* $\operatorname{spt}\psi \subset \Omega_s \subset \Omega$

$$\|\phi\psi\|_{1,p,\Omega} \le 2\left[\left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{1,p,\Omega} + \left(\sup_{\Omega}|\psi|\right)\|\phi\|_{1,p,\Omega}\right]. \tag{2.2}$$

(b) *If $m \ge 2$ and $n \ge p$ then for all $\phi, \psi \in W^{m,p}(\Omega)$ that satisfy* $\operatorname{spt}\psi \subset \Omega_s \subset \Omega$

$$\|\phi\psi\|_{m,p,\Omega} \le K\left[\left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{m,p,\Omega} + \Big(\|\psi\|_{m-1,q,\Omega} + \|\psi\|_{m-1,p,\Omega}\Big)\|\phi\|_{m,p,\Omega}\right]. \tag{2.3}$$

(c) *If $m \ge 2$ and $p > n$ then for all $\phi, \psi \in W^{m,p}(\Omega)$ that satisfy* $\operatorname{spt}\psi \subset \Omega_s \subset \Omega$

$$\|\phi\psi\|_{m,p,\Omega} \le K\left[\left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{m,p,\Omega} + \|\psi\|_{m-1,p,\Omega}\|\phi\|_{m,p,\Omega}\right]. \tag{2.4}$$

Remark. When Ω has finite volume one can combine the term $\|\psi\|_{m-1,p,\Omega}$ with its upper bound $\|\psi\|_{m-1,q,\Omega}$ in (2.3), however, the constant K will then depend on the volume of the region.

Proof of Theorem 2(a). To simplify the notation we drop the Ω, but leave the Ω_s, on the appropriate norms. To prove (2.2) we first note that, since $p > n$, without loss of generality we may assume, by the Sobolev imbedding theorem, that $\phi, \psi \in W^{1,p}(\Omega) \cap C_B(\Omega)$. Next, $\|\phi\psi\|_{1,p} \leq \|\phi\psi\|_{0,p} + \|\nabla(\phi\psi)\|_{0,p}$ and, in view of (2.1),

$$\|\phi\psi\|_{0,p} \leq \left(\sup_{\Omega_s} |\phi|\right) \|\psi\|_{1,p} .$$

However, $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$ so that, with the aid of (2.1),

$$\begin{aligned} \|\nabla(\phi\psi)\|_{0,p} &\leq \|\phi\nabla\psi\|_{0,p} + \|\psi\nabla\phi\|_{0,p} \\ &\leq \left(\sup_{\Omega_s} |\phi|\right) \|\psi\|_{1,p} + \left(\sup_{\Omega} |\psi|\right) \|\phi\|_{1,p} , \end{aligned}$$

which proves (2.2). □

Proof of Theorem 2(b) when $[\![\frac{n}{p}]\!] = m-1$ *and* $m \geq 2$. Note that $n \geq p$. Let $q \in (\frac{n}{m-1}, \frac{np}{n-p}]$ if $n > p$ and $q \in (\frac{n}{m-1}, \infty)$ if $n = p$. To prove (2.3) when $[\![\frac{n}{p}]\!] = m-1$ we first note for future reference that

$$q > p, \qquad q(m-1) > n, \tag{2.5}$$

and if $m > 2$ then $n > p$ and

$$\frac{np}{n-p} \leq \frac{n}{m-2}. \tag{2.6}$$

Now, let $\phi, \psi \in W^{m,p}(\Omega)$. Then, since $mp > n$, $\phi, \psi \in C_B(\Omega)$ by the Sobolev imbedding theorem, while $q \in [p, p_1]$ (or $q \in [p, \infty)$) yields $\psi \in W^{m-1,q}(\Omega)$ by the Gagliardo-Nirenberg-Sobolev inequality. Next,

$$\begin{aligned} \|\phi\psi\|_{m,p} &\leq \sum_{|\alpha|\leq m} \|D^\alpha(\phi\psi)\|_{0,p} \\ &= \sum_{|\alpha|\leq m} \left\| \sum_{|\beta|+|\gamma|=|\alpha|} c_{\beta\gamma} (D^\beta\phi)(D^\gamma\psi) \right\|_{0,p} \\ &\leq K \sum_{|\alpha|\leq m} \sum_{|\beta|+|\gamma|=|\alpha|} \left\|(D^\beta\phi)(D^\gamma\psi)\right\|_{0,p} , \end{aligned} \tag{2.7}$$

where $K := \max c_{\beta\gamma}$ only depends on n and m.

If $|\beta| = 0$ then $|\gamma| \leq m$ and hence by (2.1)

$$\left\|\phi(D^\gamma\psi)\right\|_{0,p} \leq \|\phi\|_{\infty,\Omega_s} \left\|D^\gamma\psi\right\|_{0,p} \leq \|\phi\|_{\infty,\Omega_s} \|\psi\|_{m,p} . \tag{2.8}$$

If $|\beta| = m$ then $|\gamma| = 0$ and hence by the Sobolev imbedding theorem and $(2.5)_2$

$$\left\|(D^\beta\phi)\psi\right\|_{0,p} \leq \left\|D^\beta\phi\right\|_{0,p} \|\psi\|_\infty \leq K \|\phi\|_{m,p} \|\psi\|_{m-1,q} . \tag{2.9}$$

If $|\gamma| = m-1$ (and $|\beta| \neq 0$) then $|\beta| = 1$. Define q' so that $\frac{1}{q} + \frac{1}{q'} = \frac{1}{p}$ and $p_{m-1} := pn/(n-(m-1)p)$ if $n > (m-1)p$ and $p_{m-1} := \infty$ otherwise. Then, by $(2.5)_2$, $q' \in (p, p_{m-1})$ and hence by Hölder's inequality and the Gagliardo-Nirenberg-Sobolev inequality ($k = m-1$)

$$\left\|(D^\beta\phi)(D^\gamma\psi)\right\|_{0,p} \le \left\|D^\beta\phi\right\|_{0,q'}\left\|D^\gamma\psi\right\|_{0,q} \le \|\phi\|_{1,q'}\,\|\psi\|_{m-1,q} \le K\,\|\phi\|_{m,p}\,\|\psi\|_{m-1,q}\,. \tag{2.10}$$

Finally, if $2 \le |\beta| \le m-1$ then $|\gamma| \le m-2$. Note $|\beta| + |\gamma| \le m$, $m \ge 3$, and $n \neq p$ (since $n/p \ge m-1 \ge 2$). Thus, by Hölder's inequality,

$$\begin{aligned}\left\|(D^\beta\phi)(D^\gamma\psi)\right\|_{0,p} &\le \left\|D^\beta\phi\right\|_{0,\frac{pn}{n-(m-|\beta|)p}}\left\|D^\gamma\psi\right\|_{0,\frac{pn}{(m-|\beta|)p}} \\ &\le \|\phi\|_{|\beta|,\frac{pn}{n-(m-|\beta|)p}}\,\|\psi\|_{m-|\beta|,\frac{pn}{(m-|\beta|)p}}\,. \end{aligned} \tag{2.11}$$

Then, in view of the Gagliardo-Nirenberg-Sobolev inequality ($k = m-|\beta|$ and $k = |\beta|-1$),

$$\|\phi\|_{|\beta|,\frac{pn}{n-(m-|\beta|)p}} \le K\,\|\phi\|_{m,p}\,, \quad \|\psi\|_{m-|\beta|,\frac{n}{m-|\beta|}} \le K\,\|\psi\|_{m-1,q}\,, \tag{2.12}$$

since $q \in (\frac{n}{m-1}, \frac{n}{m-2}]$ by (2.6) and the definition of q. The desired result, (2.3), now follows in the case when $\llbracket\frac{n}{p}\rrbracket = m-1$ from (2.7)–(2.12). □

Proof of Theorem 2(b) when $1 \le \llbracket\frac{n}{p}\rrbracket < m-1$. We prove (2.3) by induction on m. Note that $n \ge p$. Define $\widehat{m} := \Upsilon + 1 = \llbracket\frac{n}{p}\rrbracket + 1$. Then $\widehat{m} > n/p \ge \widehat{m}-1$ and $\widehat{m} \ge 2$. The induction starts at $m = \widehat{m}$. Then, as we have just proven, (2.3) is valid for $m = \widehat{m}$ and any q in the appropriate interval. To continue the induction we assume (2.3) is valid, for some $m \ge \widehat{m}$ and q, and show it is valid for $m+1$ and the same q.

Let $\phi, \psi \in W^{m+1,p}(\Omega)$. Then $q \in (\frac{n}{\Upsilon}, \frac{np}{n-p}]$ if $n > p$ and $q \in (\frac{n}{\Upsilon}, \infty)$ if $n = p$; consequently $\psi \in W^{m,q}(\Omega)$ by the Gagliardo-Nirenberg-Sobolev inequality. We again note $\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$ and hence

$$\begin{aligned}\|\phi\psi\|_{m+1,p} &\le \|\phi\psi\|_{m,p} + \|\nabla(\phi\psi)\|_{m,p} \\ &\le \|\phi\psi\|_{m,p} + \|\psi\nabla\phi\|_{m,p} + \|\phi\nabla\psi\|_{m,p}\,. \end{aligned} \tag{2.13}$$

However, by the induction hypothesis

$$\begin{aligned}\|\phi\psi\|_{m,p} &\le K\left[\left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{m,p} + \Big(\|\psi\|_{m-1,q} + \|\psi\|_{m-1,p}\Big)\|\phi\|_{m,p}\right] \\ &\le K\left[\left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{m+1,p} + \Big(\|\psi\|_{m,q} + \|\psi\|_{m,p}\Big)\|\phi\|_{m+1,p}\right], \end{aligned} \tag{2.14}$$

and, similarly,

$$\|\phi\nabla\psi\|_{m,p} \le K\left[\left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{m+1,p} + \left(\|\psi\|_{m,q} + \|\psi\|_{m,p}\right)\|\phi\|_{m+1,p}\right], \tag{2.15}$$

$$\|\psi\nabla\phi\|_{m,p} \le K\left[\left(\sup_{\Omega}|\nabla\phi|\right)\|\psi\|_{m,p} + \left(\|\psi\|_{m,q} + \|\psi\|_{m,p}\right)\|\phi\|_{m+1,p}\right], \tag{2.16}$$

since $\Omega_s \subset \Omega$, while by the Sobolev imbedding theorem ($k = m$)

$$\sup_{\Omega}|\nabla\phi| \le K\,\|\phi\|_{m+1,p}\,. \tag{2.17}$$

Equation (2.3) with m replaced by $m+1$, now follows from (2.13)–(2.17). □

Proof of Theorem 2(c). Note that $p > n$. First we prove (2.4) for $m = 2$. Let $\phi, \psi \in W^{2,p}(\Omega)$. Then (2.13) with $m = 1$ is valid, however, (2.14)–(2.16) must be replaced by (see (2.2))

$$\begin{aligned}\|\phi\psi\|_{1,p} &\le 2\left[\left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{1,p} + \left(\sup_{\Omega}|\psi|\right)\|\phi\|_{1,p}\right],\\ \|\psi\nabla\phi\|_{1,p} &\le 2\left[\left(\sup_{\Omega}|\nabla\phi|\right)\|\psi\|_{1,p} + \left(\sup_{\Omega}|\psi|\right)\|\nabla\phi\|_{1,p}\right],\end{aligned} \tag{2.18}$$

and an appropriate estimate for $\|\phi\nabla\psi\|_{1,p}$. Clearly, $\|\cdot\|_{1,p} \le \|\cdot\|_{2,p}$ and since $p > n$ the Sobolev imbedding theorem ($k = 1$) yields

$$\sup_{\Omega}|\psi| \le K\,\|\psi\|_{1,p}\,, \quad \sup_{\Omega}|\nabla\phi| \le K\,\|\nabla\phi\|_{1,p} \le K\,\|\phi\|_{2,p}\,. \tag{2.19}$$

Thus, we need only estimate the term $\|\phi\nabla\psi\|_{1,p}$, which replaces (2.15), to finish the proof when $m = 2$.

We note $\nabla(\phi\nabla\psi) = \phi\nabla\nabla\psi + \nabla\psi \otimes \nabla\phi$ and hence

$$\begin{aligned}\|\phi\nabla\psi\|_{1,p} &\le \|\phi\nabla\psi\|_{0,p} + \|\nabla(\phi\nabla\psi)\|_{0,p}\\ &\le \|\phi\nabla\psi\|_{0,p} + \|\phi\nabla\nabla\psi\|_{0,p} + \|\nabla\psi\otimes\nabla\phi\|_{0,p}\,.\end{aligned} \tag{2.20}$$

In view of (2.1) the first two terms on the right-hand side of (2.20) satisfy

$$\|\phi\nabla\psi\|_{0,p} \le \left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{2,p}\,, \quad \|\phi\nabla\nabla\psi\|_{0,p} \le \left(\sup_{\Omega_s}|\phi|\right)\|\psi\|_{2,p}\,, \tag{2.21}$$

while the last term on the right-hand side of (2.20) satisfies

$$\|\nabla\psi\otimes\nabla\phi\|_{0,p} \le \left(\sup_{\Omega}|\nabla\phi|\right)\|\psi\|_{1,p}\,. \tag{2.22}$$

Equation (2.4) with $m = 2$ now follows from (2.13) with $m = 1$ and (2.18)–(2.22).

To complete the proof we note that (2.4) can be obtained by induction on m for $m \geq 2$. However, the required steps are identical to those in the proof of Theorem 2(b), when $1 \leq [\![\frac{n}{p}]\!] < m - 1$, with the term $\|\psi\|_{m,q}$ deleted. $\square$

Remark. If $m \geq 2$, $n \geq p$, and $\Omega_s \subset \Omega$ is open then one can show that, for all $\phi, \psi \in W^{m,p}(\Omega)$ that satisfy $\operatorname{spt}\psi \subset \Omega_s \subset \Omega$,

$$\|\phi\psi\|_{m,p,\Omega} \leq K \left(\sum_{k=0}^{m-\widehat{m}} \left[\|\phi\|_{C_B^k(\Omega_s)} \|\psi\|_{m-k,p,\Omega} \right] + \|\psi\|_{m-1,q,\Omega} \|\phi\|_{m,p,\Omega} \right), \tag{2.23}$$

where $m \geq \widehat{m} = \Upsilon + 1 := [\![\frac{n}{p}]\!] + 1$ and

$$\|\phi\|_{C_B^k(\Omega_s)} := \sum_{|\alpha| \leq k} \sup_{\mathbf{x} \in \Omega_s} |D^\alpha \phi(\mathbf{x})|.$$

Equation (2.23) is obtained by induction on m for $m \geq 2$. The initial step is the above proof of Theorem 2(b) when $[\![\frac{n}{p}]\!] = m - 1$. The induction is then similar to that presented above in the proof of Theorem 2(b) when $1 \leq [\![\frac{n}{p}]\!] < m - 1$. The only significant difference is that one does not use the estimate (2.17), but instead leaves the appropriate version of (2.16) as it is, since each step in the induction argument will now add an additional derivative to the ϕ term that multiplies $\|\psi\|_{m-k,p}$.

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