To solve an exact eqn., the key is to find ψ:

$\because \psi_x = M, \ \psi_y = N.$

$\therefore \ \psi_x = M \Rightarrow \int \psi_x \, dx = \int M dx \Rightarrow \psi = \int M dx + h(y)$, (holding y constant).

[Here $h(y)$ plays the role of the arbitrary const.].

Thus $\psi_y = \frac{\partial}{\partial y} \int M dx + \frac{d}{dy} h(y) = N.$

i.e. $\frac{d}{dy} h(y) = N - \frac{\partial}{\partial y} \int M dx \Rightarrow h(y) = \int [N - \frac{\partial}{\partial y} \int M dx] \, dy.$

Ex: $(y \cos x + 2xe^y) + (\sin x + x^2 e^y - 1) y' = 0.$

Here: $M = y\cos x + 2xe^y$, $N = \sin x + x^2 e^y - 1$.

check: $M_y = \cos x + 2xe^y$, $N_x = \cos x + 2xe^y$.

$\therefore$ it is an exact eqn. ($\because M_y = N_x$).

Thus $\exists \ \psi$ s.t. $\psi_x = M, \ \psi_y = N.$

Integrating $\psi_x = M$: $\quad \psi = \int (y\cos x + 2xe^y) dx + h(y)$.

$\quad = y \sin x + x^2 e^y + h(y)$

Differentiate it: $\psi_y = \sin x + x^2 e^y + \frac{d}{dy} h(y)$, which is equal to N. That is:

$\sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1.$

i.e. $h'(y) = -1, \Rightarrow h(y) = -y.$

Thus $\psi(x,y) = y \sin x + x^2 e^y - y = c$ gives the general solution IMPLICITLY.

Ex: (§2.6-1) Find b s.t.

$(ye^{2xy} + x)dx + bxe^{2xy}dy = 0$. is an exact eqn. Solve it.

$M = ye^{2xy} + x$, $N = bxe^{2xy}$.

To make the ODE be exact, we need $M_y = N_x$. (i.e. $\frac{\partial}{\partial y}M = \frac{\partial}{\partial x}N$).

$\because$ $M_y = 2xye^{2xy} + e^{2xy}$, $N_x = be^{2xy} + 2bxye^{2xy}$

$\therefore$ $b = 1$, then $M_y = N_x$.

Solve the ODE when $b = 1$.

$$\psi_x = M . \Rightarrow \psi = \int M dx + h(y)$$
$$= \int (ye^{2xy} + x) dx + h(y)$$
$$= \frac{1}{2}x^2 + \frac{1}{2}e^{2xy} + h(y).$$

Thus $\psi_y = xe^{2xy} + \frac{d}{dy}h(y) = N$.

i.e. $xe^{2xy} + h'(y) = xe^{2xy}$

$h'(y) = 0$, $\Rightarrow h = c$.

choosing $h = 0$, then

$$\psi(x, y) = \frac{1}{2}x^2 + \frac{1}{2}e^{2xy} = c$$

gives the general solution of the ODE implicitly.

Transfer an ODE into an exact eqn. when

either $\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$ depends on only x, or $\frac{\mu_y}{\mu} = -\frac{M_y - N_x}{M}$ depends on only y.

Ex (§2.6: 28) Solve $y\,dx + (2xy - e^{-2y})\,dy = 0$.

Here $M = y$, $N = 2xy - e^{-2y}$.

$M_y = 1$, $N_x = 2y$. Thus, the ODE is not exact.

check (i) $\frac{M_y - N_x}{N} = \frac{1-2y}{2xy - e^{-2y}}$, depends on both x and y.

(ii) $-\frac{M_y - N_x}{M} = -\frac{1-2y}{y}$ depends on only y.

Then solve $\mu_y = -\frac{1-2y}{y}\mu$ for μ.

$$\ln \mu = -\int \frac{1-2y}{y}\,dy = -\ln y + 2y.$$

$$\mu = y^{-1} e^{2y}.$$

Multiply the original ODE:

$$e^{2y}\,dx + (2xe^{2y} - y^{-1})\,dy = 0$$

Now: $M = e^{2y}$, $N = 2xe^{2y} - y^{-1}$,

$M_y = 2e^{2y}$, $N_x = 2e^{2y}$. Thus, the new ODE is an exact eqn.

Set $\psi_x = M$, $\Rightarrow \psi = \int e^{2y}\,dx + h(y) \Rightarrow \psi = xe^{2y} + h(y)$.

$\Rightarrow \psi_y = 2xe^{2y} + h'(y) = N$, i.e., $2xe^{2y} + h' = 2xe^{2y} - y^{-1}$.

$\Rightarrow h' = -y^{-1}$, $\Rightarrow h = -\ln|y|$.

Thus $\psi = xe^{2y} - \ln|y| = C$ gives the general solution implicitly.